Codes for Simultaneous Transmission of Quantum and Classical Information

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**Advantage** compared to independent solutions?

For the finite length case: in Kremsky et al. [2008], the authors consider the problem in the context of so-called entanglement-assisted codes. The examples given in Kremsky et al. [2008], e.g. $[[9, 1 : 2, 3]]$, however, fail to demonstrate an advantage compared to stabilizer quantum codes. (Even $[[8, 3, 3]]$ exists)
Introduction

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(a) Codeword stabilized (CWS) codes Cross et al. [2009]
(b) Union stabilizer codes Grassl and Rötteler [2008]

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- A general construction $\Rightarrow$ up to 34 qubits. (See arXiv version: 1701:06963)
Introduction

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- A general construction ⇒ up to 34 qubits. (See arXiv version: 1701:06963)
- Linear program bound on $n, k, m, d$
Our discussion is based on the theory of stabilizer quantum codes and its connection to classical error-correcting codes (see, e.g., Calderbank et al. [1998]). We use the following notations.

- \((n, K, d)\)_{q}
- \([n, k, d]\)_{q}
- \((n, M, d)\)_{q}
- \([n, m, d]\)_{q}
- \([n, k:m, d]\)_{q}
- \(((n, K:M, d))\)_{q}
Trivial Construction

\[(n, KM, d)_q \Rightarrow ((n, K: M, d))_q\]
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\(((n, KM, d))_q \Rightarrow ((n, K:M, d))_q\)

\([n, k:m, d]_q \Rightarrow [n, k - 1:m + 1, d]_q\)

Our goal is to find codes that have better parameters than the codes that can be obtained by these trivial constructions.
Trivial Construction

- \((n, KM, d))_q \Rightarrow ((n, K:M, d))_q\)
- \([n, k:m, d]_q \Rightarrow [n, k - 1:m + 1, d]_q\)
- \([n_1, k_1, d]_q + [n_2, m_2, d]_q \Rightarrow [n_1 + n_2, k_1:m_2, d]_q\)

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A hybrid quantum code $\mathcal{C} = ((n, K:M))_q$ can be described by a collection

$$\{\mathcal{C}^{(\nu)} : \nu = 1, \ldots, M\}$$

of $M$ quantum codes $\mathcal{C}^{(\nu)} = ((n, K, d))_q$. The classical information $\nu$ determines which quantum code $\mathcal{C}^{(\nu)}$ is used to encode the quantum information.

In the following, we will use Greek letters when referring to classical information. Assume that $\{|c_i^{(\nu)}\rangle : i = 1, \ldots, K\}$ is an orthonormal basis for the code $\mathcal{C}^{(\nu)}$. 
In order to be able to correct the linear span of error operators \( \{ E_k : k = 1, 2, \ldots \} \), each of the codes \( C^{(\nu)} \) has to obey the Knill-Laflamme conditions Knill and Laflamme [1997], i. e.,

\[
\langle c_i^{(\nu)} | E_k^\dagger E_\ell | c_j^{(\nu)} \rangle = \alpha_{k\ell}^{(\nu)} \delta_{ij}.
\]
Error Correction Conditions

In order to be able to correct the linear span of error operators \( \{ E_k : k = 1, 2, \ldots \} \), each of the codes \( C^{(\nu)} \) has to obey the Knill-Laflamme conditions [Knill and Laflamme 1997], i.e.,

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Note that the constants \( \alpha_{k\ell}^{(\nu)} \in C \) may depend on the classical information \( \nu \). To retrieve the classical information \( \nu \), one has to be able to perfectly distinguish the states \( |c_i^{(\nu)}\rangle \) and \( |c_j^{(\mu)}\rangle \) for \( \nu \neq \mu \) and arbitrary \( i \) and \( j \) after an error.

\[
\langle c_i^{(\nu)} | E_k^{\dagger} E_\ell | c_j^{(\mu)} \rangle = 0, \text{ for } \mu \neq \nu.
\]
A hybrid quantum code $\mathcal{C} = ((n, K:M))_q$ with orthonormal basis states $\{|c_i^{(\nu)}\rangle: i = 1, \ldots, K, \nu = 1, \ldots, M\}$ can correct all errors $\{E_k: k = 1, 2, \ldots\}$ if and only if
\[
\langle c_i^{(\nu)}|E_k^\dagger E_\ell|c_j^{(\mu)}\rangle = \alpha_{k\ell}^{(\nu)} \delta_{ij} \delta_{\mu \nu}.
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Theorem

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Discussions:

- When \( \alpha_{k\ell}^{(\nu)} \) do not depend on \( \nu \), condition reduces to Knill-Laflamme condition for a quantum code \( \mathcal{C} = ((n, KM))_q \).
- For hybrid codes with better parameters, there should be at least a pair \( \nu, \mu \) and errors \( E_k, E_\ell \) such that \( \alpha_{k\ell}^{(\nu)} \neq \alpha_{k\ell}^{(\mu)} \).
Error Correction Conditions

**Theorem**

A hybrid quantum code \( C = ((n, K:M))_q \) with orthonormal basis states \( \{|c_i^{(\nu)}\rangle : i = 1, \ldots, K, \nu = 1, \ldots, M\} \) can correct all errors \( \{E_k : k = 1, 2, \ldots\} \) if and only if

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- For hybrid codes with better parameters, there should be at least a pair \( \nu, \mu \) and errors \( E_k, E_\ell \) such that \( \alpha_{k\ell}^{(\nu)} \neq \alpha_{k\ell}^{(\mu)} \).

- When the error operators \( E_k \) are unitary, \( \alpha_{kk}^{(\nu)} = 1 \). Then \( \alpha_{k\ell}^{(\nu)} \neq 0 \) for some \( \nu \) and \( k \neq \ell \), which suggests that some of the codes \( C^{(\nu)} \) might be taken to be degenerate codes.
We outline the construction of hybrid quantum codes in the framework of CWS codes/union stabilizer codes. We start with a quantum code $C^{(0)} = ((n, K, d))_q$ which is a CWS code that might even be a stabilizer code $C^{(0)} = [n, k, d]_q$. 
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$$C^{(\nu)} = t_\nu C^{(0)}$$

with $\{t_\nu : \nu = 1, \ldots M\}$ a set of $M$ translation operators.
Code Construction

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(a) $S \Rightarrow$ self orthogonal classical code $C_0$.

(b) $C_0 \subseteq C^*_0 \Rightarrow N$

$$d = \min \{ \text{wgt } c : c \in C^*_0 \setminus C_0 \} > \min \{ \text{wgt } c : c \in C^*_0 \setminus \{0\} \}.$$  

The codes $C^{(\nu)} = t_\nu C^{(0)}$ are associated with cosets $C^*_0 + t_\nu$ of the normalizer code $C^*_0$. 

When the cosets $C_0^* + t_\nu$ and $C_0^* + t_\mu$ are different, then the codes $C^{(\nu)}$ and $C^{(\mu)}$ will be orthogonal to each other. The hybrid code $C$ is associated with the classical code

$$C^* = \bigcup_{\nu=1}^{M} C_0^* + t_\nu.$$ 

When the union of the codes is an additive code, the hybrid quantum code will be a stabilizer code.
Note that, in general, we have the chain of classical codes

\[ C \leq C_0 \leq C_0^* \leq C^*. \]
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The minimum distance of the quantum code associated with \( C^* \) is computed as

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Note that the minimum(\( d \)) is taken over a smaller set compared to \( d' \), as \( C \leq C_0 \), and hence \( d \geq d' \).
In summary, we have the following construction.

**Theorem**

Let $C_0 = (n, q^{n-k}, d_0)_{q^2}$ be a classical additive code that is contained in its symplectic dual $C_0^*$. Further, let $C^* = (n, q^{n+k+m}, d')_{q^2}$ be an additive code containing $C_0^*$. Then there exists a hybrid stabilizer code $C = [n, k:m, d]_q$ encoding $k$ qudits and $m$ classical symbols. The minimum distance of $C$ is given by

$$d = \min\{\text{wgt } c : c \in C^* \setminus C_0\}.$$
In order to obtain bounds on the parameters of hybrid stabilizer codes \([n, k:m, d]_q\), we consider the homogeneous weight enumerators of the associated code \(C_0\) and its symplectic dual \(C_0^*\), as well as the code \(C^*\) and its symplectic dual \(C^*_0\):

\[
\mathcal{W}_{C_0}(X, Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w, \quad \mathcal{W}_{C_0^*}(X, Y) = \sum_{w=0}^{n} A_w X^{n-w} Y^w, \\
\mathcal{W}_{C}(X, Y) = \sum_{w=0}^{n} B_w X^{n-w} Y^w, \quad \mathcal{W}_{C^*}(X, Y) = \sum_{w=0}^{n} B_w X^{n-w} Y^w.
\]
The weight enumerators of $C_0$ and $C_0^*$, as well as those of $C$ and $C^*$, are related by the MacWilliams transformation, i.e.,

$$\mathcal{W}_{C_0^*}(X, Y) = \frac{1}{|C_0|} \mathcal{W}_{C_0} \left( X + (q^2 - 1)Y, X - Y \right),$$

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Nestedness of the codes implies the condition

$$0 \leq B_w^\perp \leq A_w^\perp \leq A_w \leq B_w,$$

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When the hybrid code has minimum distance $d$, we have

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More details can be found in the proceedings, including tables.
Results (Code Search)

Search for $\mathcal{C} = \left[ n, k:m, d \right]_2$ codes with distance $d \geq 3$.

- **Union Stabilizer:**
  1. Start with the self-dual codes from the classification in Danielsen, Danielsen and Parker [2006].
  2. Construct impure quantum codes $\left[ n, 1, d \right]_2$ Then look for additional vectors for the encoding of classical information, resulting in an $\left[ n, 1:m', d \right]_2$ hybrid code.
  3. In some cases, the code $\left[ n, 1:m', d \right]_2$ is in fact a $\left[ n, k:m' - k + 1, d \right]_2$. 
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  3. In some cases, the code $[n, 1:m', d]_2$ is in fact a $[n, k:m' - k + 1, d]_2$.

- **CWS Framework:**
  (a) start with the graph state from the classification in Danielsen, Danielsen and Parker [2006].
  (b) Construct impure code using CWS framework, then look for additional vectors for the encoding of classical information by searching for MAX-Clique. Results in a hybrid code with parameters $[n, k:m'', d]_2$
  (c) $\prod_i E_k^\dagger E_i \prod_j = 0, i \neq j$
Results

Theorem

There exist hybrid codes with the following parameters:

\[
\begin{align*}
[7, 1:1, 3]_2, & \quad [9, 2:2, 3]_2, & \quad [10, 3:2, 3]_2, & \quad [11, 4:2, 3]_2, \\
[11, 1:2, 4]_2, & \quad [13, 1:4, 4]_2, \\
[13, 1:1, 5]_2, & \quad [14, 1:2, 5]_2, & \quad [15, 1:3, 5]_2, \\
[19, 9:1, 4]_2, & \quad [20, 9:2, 4]_2, & \quad [21, 9:3, 4]_2, & \quad [22, 9:4, 4]_2 \cdots
\end{align*}
\]

All these codes have better parameters than codes obtained from the best quantum codes using trivial construction.
Results (Seven qubits)

No $[7,2,3]_2$

Starting with this impure code, we obtain a hybrid code with parameters $[7, 1:1, 3]_2$.

The additional generator that is used to encode one classical bit is given below the double horizontal line.

We have not found a $[7, 1:2, 3]_2$ which is not ruled out by linear programming.
Results (Eight qubits)

- For eight qubits, there is a quantum code with parameters $[8, 3, 3]_2$. Using trivial construction, we obtain an optimal hybrid code with parameters $[8, 2:1, 3]_2$, as well as a code $[8, 1:2, 3]_2$.

- We have not found a hybrid code with parameters $[8, 1:3, 3]_2$ that might exist.
Results (Nine qubits)

For nine qubits, we found a hybrid code $[[9, 2:2, 3]]_2$

Taking all possible products of the two generators below the double horizontal line we obtain the four translation operators $t^{(1)} = id, t^{(2)}, t^{(3)},$ and $t^{(4)} = t^{(2)}t^{(3)}$ used to encode two extra classical bits.
Results (10 qubits)

- A hybrid code $[10, 3:2, 3]_2$ exists.
- Via linear programming it is found that this code is optimal in the sense that it encodes the maximal possible number $m$ of additional classical bits among all codes $[10, 3:m, 3]_2$. 
The first non-trivial hybrid code with distance $d = 4$ has been found for eleven qubits. A hybrid code $[[11, 1:2, 4]]_2$ is given. We found a hybrid code $[[11, 4:2, 3]]_2$ as well.
Appending two qubits in the state $|0\rangle$ to the impure quantum code $[[11, 1, 4]]_2$ given above the double horizontal line, one obtains an impure code $[[13, 1, 4]]_2$. This code can additionally transmit four classical bits, i.e., one obtains the hybrid code $[[13, 1:4, 4]]_2$. 
We generalize this construction by following theorem.

**Theorem**

Let $C_1 = [[n, k_1, d_1]]_q \subset C_2 = [[n, k_2, d_2]]_q$ be nested quantum codes. Further, let $C_3 = [[n_3, k_2 - k_1, d_3]]_q$ be a classical linear code. Then there is a hybrid quantum code $C = [[n + n_3, k_1:(k_2 - k_1), d]]_q$ with $d \geq \min(d_1, d_2 + d_3)$.

From the nested stabilizer codes $[[11, 1, 5]]_2 \subset [[11, 4, 3]]_2$ and classical codes $[[n_3, n_3 - 1, 2]]_2$, one obtains hybrid codes $[[13, 1:1, 5]]_2$, $[[14, 1:2, 5]]_2$, and $[[15, 1:3, 5]]_2$. Similarly, from $[[17, 9, 4]]_2 \subset [[17, 13, 2]]_2$, one gets $[[19, 9:1, 4]]_2$, $[[20, 9:2, 4]]_2$, $[[21, 9:3, 4]]_2$, and $[[22, 9:4, 4]]_2.$
The code conditions derived here suggest that one should start with good impure quantum codes.

In order to find a direct construction of hybrid codes with good parameters, a first step could be to develop methods to construct good non-trivial impure codes.

How?
Conclusions

- We consider the characterization as well as the construction of quantum codes that allow to transmit both quantum and classical information, which we refer to as “hybrid codes”.
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- We construct hybrid codes $[[n, k:m, d]]_q$ with length $n$ and distance $d$, that simultaneously transmit $k$ qudits and $m$ symbols from a classical alphabet of size $q$. 
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Many good codes up to 34 qubits have been found. All these codes have better parameters than hybrid codes obtained from the best known stabilizer quantum codes.
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Thank you!
Questions/Answers
Additionally, we have:

\[ A_0 = A_0 = B_0 = 1, \]

\[ \sum_{w=0}^{n} A_w = q^{n-k}, \quad \sum_{w=0}^{n} A_w = q^{n+k}, \]

\[ \sum_{w=0}^{n} B_w = q^{n-k-m}, \quad \sum_{w=0}^{n} B_w = q^{n+k+m}. \]
Additionally, we have:

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When a hybrid stabilizer code \([n, k:m, d]\)^q exists, the linear program for the variables \(B_w^\perp, A_w^\perp, A_w, \text{ and } B_w\) has an integer solution.
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When a hybrid stabilizer code \([n, k:m, d]_q\) exists, the linear program for the variables \(B_w^\perp, A_w^\perp, A_w,\) and \(B_w\) has an integer solution. For qubit codes, we can strengthen the LP by additionally considering the shadow enumerator Rains [1999]

\[ S_{C_0}(X, Y) = \frac{1}{|C_0|} \mathcal{W}_{C_0} \left( X + (q^2 - 1) Y, Y - X \right), \]

which has to have non-negative integer coefficients. Ref to Rains.
Using CPLEX V12.6.3.0, we checked whether the integer program is feasible. More precisely,

- we first fix the length $n$, number of qudits $k$, and number $M = 2^m$ of classical symbols.
- Then we look for the largest minimum distance $d$ for which the integer program is found to be feasible.
- The resulting bounds on the parameters $[n, k:m, d]_2$ are listed in Table, i.e., for fixed parameters $n$, $k$, and $d$, the largest possible value for $m$ is given.
- For $n > 14$, there seem to be some precision issues, so we list only the bounds for $n \leq 14$. 
### LP Bound ($d = 3$)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
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