

The Three-Object Ising Twist: Channels and Formal Scope

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Verdict: scope-restriction (open).

1 The construction and its four channels

The Ising fusion rules and F -symbols are given in Ref. [BMW⁺17], Appendix D.2. The weighted bond-object construction is a project example motivated by the question following Theorem 4.14 of Ref. [CPGSV16]; it is not a tensor printed in either paper. Its local construction record is `Notes/OpenProblems TN/checks/asym_ising_action_data.md`, sections 0 and 1.4. The relevant equations are stated here independently of that local record.

Let A_1, A_ψ, A_σ be the three Ising operator tensors, of auxiliary dimensions 3, 3, 4. Write $\hat{A}_\sigma = \sqrt{2}A_\sigma$ and $\Theta_3 = 5A_1 \oplus 3A_\psi \oplus 2A_\sigma$. On a positive-length ring, $O_L(A_1)O_L(\hat{A}_\sigma) = O_L(\hat{A}_\sigma)$, $O_L(A_\psi)O_L(\hat{A}_\sigma) = O_L(\hat{A}_\sigma)$, and $O_L(\hat{A}_\sigma)^2 = 2^L(O_L(A_1) + O_L(A_\psi))$. Hence

$$O_L(\Theta_3)O_L(\hat{A}_\sigma) = (5^L + 3^L)O_L(\hat{A}_\sigma) + (2\sqrt{2})^L(O_L(A_1) + O_L(A_\psi)). \quad (1)$$

The same formula holds for traces along every nonempty word. It supplies four weighted normal blocks, namely $5\hat{A}_\sigma$, $3\hat{A}_\sigma$, $2\sqrt{2}A_1$, and $2\sqrt{2}A_\psi$. Asymmetric compression gives $40 = 4 + 4 + 3 + 3 + 26$, with twenty-six one-dimensional zero diagonal blocks in an upper-triangular presentation. An explicit change of auxiliary coordinates strengthens this to a block-diagonal presentation. It is the direct sum of the twenty-four-dimensional gauge for the two abelian bond objects and the sixteen-dimensional gauge for $\sigma \otimes \sigma$. The inverse of the latter is half its transpose. After separating the four active blocks from the twenty-six zero coordinates, the off-diagonal remainder vanishes. The left projection and right inclusion intertwine every letter with its weighted target. This statement concerns this explicit choice; the compression obtained only from the trace theorem need not be the same choice.

The three-term coefficient $5^L + 3^L + 2^L$ for a single σ target, and the three-slot compression requested in issue 7833, are not the construction's fusion rule. The last bond object contributes to 1 and ψ through $\sigma \times \sigma = 1 + \psi$. For example, at the one-letter word $(h, h') = (0, 0)$, the trace of the stacked tensor

is $2\sqrt{2}$, whereas the trace of $\hat{A}_\sigma^{00}$ is zero. This already excludes the proposed single-target formula.

2 The full boundary tensors remain distinct

At the numerical point $(\lambda_1, \lambda_\psi, \lambda_\sigma) = (1/2, 3/10, 1/5)$, the unrescaled bond object is $\Theta_3/10$. This scalar relation concerns the local forty-dimensional product above. The full round-45B boundary tensor of flag f has site space $\mathbb{C}^4 \otimes P_f \otimes \mathbb{C}^4$, of dimension 48, 48, 64 for $f = 1, \psi, \sigma$, and horizontal indices (h, r, s) with $h \in \{0, \dots, 9\}$ and $r, s \in \{0, \dots, 3\}$. Its matrices take the form

$$M_f^{(h,r,s),(h',r',s')} = X_f^{h,r,s,h'} \otimes E_{r's'}. \quad (2)$$

Their fusion is the separate assertion

$$O_L(M_f)O_L(M_{f'}) = \sum_{b,c} \lambda_b^L N_{fbf'}^c O_L(M_c), \quad (3)$$

where $N_{fbf'}^c$ is the multiplicity of c in $f \otimes b \otimes f'$. In particular $N_{\sigma\sigma\sigma}^\sigma = 2$. Neither (1) nor a common scalar rescaling proves (3).

At $(f, f') = (\sigma, \sigma)$ the stacked auxiliary dimension is 4096, the alphabet has 25600 letters, and the active targets have dimensions 48, 48, 48, 48, 64, 64. Direct exhaustive kernel evaluation of dense matrices at this size is not a suitable extension of the existing small examples. No exact full-tensor certificate is supplied here. The elimination plan is to define the reduced letters X_f at the rational point, prove their fusion from the sparse Ising F -move identities and the register matrix-unit contraction, and then restore the outer register in (2). Positivity, normalization and refinement channels must be established for those same literal tensors. This remaining scope is the second part of <https://github.com/LionSR/TNLean/issues/7833>; it must not be marked formalized on the basis of the forty-dimensional compression.

3 Formal declarations

The local result is in `TNLean/MPS/Examples/Ising/IsingThreeObjectTwist.lean`: `IsingTwist.thetaThree_mpo_mul_isingSigma`, `IsingTwist.isingBondObjectThree_trace_evalWord`, `IsingTwist.isingBondObjectThree_compression_exists`, and `IsingTwist.isingBondObjectThree_z_eq`. The explicit split compression is in `TNLean/MPS/Examples/Ising/IsingThreeObjectGauge.lean`: `IsingTwist.isingThreeSplitCompression`, `IsingTwist.isingBondObjectThree_split_remainder`, `IsingTwist.isingBondObjectThree_mul_right_eq_right_mul`, and `IsingTwist.isingBondObjectThree_left_mul_eq_mul_left`.

References

- [BMW⁺17] Nick Bultinck, Michael Mariën, Dominic J. Williamson, Mehmet B. Şahinoğlu, Jutho Haegeman, and Frank Verstraete. Anyons and matrix product operator algebras. *Annals of Physics*, 378:183–233, 2017.
- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.