Dependency Audit for the Non-periodic MPS Fundamental Theorem Blueprint

2026-05-08

Verdict: clarification (open).

1 Scope

This note records the source-faithful answer to the organizational questions in issue 1530. It concerns the non-periodic Fundamental Theorem route based on [CPGSV16, CPGSV21], not the direct periodic theorem of [DlCCSPG17] and not the fixed-length algebraic theorem of [MGRPG$^+$18]. It also records which assumptions in the current conditional after-blocking statements are source inputs that still have to be produced, rather than conclusions already proved in the blueprint.

The guiding source passages are the BNT definition, Proposition `prop:char-BNT`, the expansion `eq:II_ABasicTensors`, the block-injective input Proposition `propblockinj`, the proportional Fundamental Theorem `thm1`, and the equal-MPV Corollary `II_cor2` in [CPGSV16, lines 271–356]. In the appendix, `Lem:app_simple` is lines 1155–1163, `thm1` is restated at lines 1167–1170 and proved at line 1182, and `II_cor2` is stated at lines 1172–1179. The global gauge formula is `eq:II_auxcor`, stated at lines 1175–1177 and obtained in lines 1189–1192. The review version is [CPGSV21, Section IV]. This note follows the same convention as the previous paper-gap notes: the source theorem is the standard of comparison, while formal helper structures are acceptable only when they are identified as helper structures.

Notation. Let d be the physical dimension. For a tensor $A = (A^i)_{i=1}^d$, write $A^{[p]}$ for its length-p blocking, whose physical dimension is d^p. For a word $w = i_1 \cdots i_L$, write $|w| = L$ and $A^w = A^{i_1} \cdots A^{i_L}$. If $(A_j)_{j=1}^g$ is a basis of normal tensors, set $\tilde{A}^i = \bigoplus_{j=1}^g A_j^i$ and $\tilde{A}^w = \tilde{A}^{i_1} \cdots \tilde{A}^{i_L}$. The tensor $\mathbf{0}_{d^p,z}$ is the zero tensor of physical dimension d^p and bond dimension z, and A_{nz} denotes the nonzero part after removing a zero direct summand. The symbol $\sim$ denotes the canonical-form direct-sum decomposition, up to the usual gauge change. In BNT comparisons, π denotes a permutation of the representative normal tensors, X_j an invertible gauge, and $\zeta_j \in \mathbb{C}^\times$ the nonzero phase factor in the derived relation $B^i_{\pi(j)} = \zeta_j X_j A^i_j X_j^{-1}$.

2 Block permutation and separation

The blueprint chapter titled "Block Permutation and Separation" contains more than one mathematical ingredient. The product-algebra part proves that an automorphism of

$$\prod_{j=1}^{g} M_{D_j}(\mathbb{C})$$

permutes the simple factors and is inner on each factor. This is the algebraic mechanism used by same-structure and block-injective comparison statements. The invariant-projection material belongs to canonical-form reduction. The block-separation lemmas prove that, under strict canonical-form hypotheses, a same-MPV identity for two block-diagonal tensors separates into same-MPV identities for the individual blocks.

For the current conditional non-periodic Fundamental Theorem statements, this chapter is not used wholesale. Its visible downstream uses are the canonical-form condition for BNT data and the same-structure comparison lemma. The latter is the special case in which the two tensors already have the same number of blocks, the same bond dimensions, and the same weights; it is not the CPSV BNT matching step. In the source proof, the general non-periodic theorem instead proceeds through BNT representatives, eventual linear independence, block-injective fixed-length span, and scalar power sums.

Thus the block-permutation chapter should stay before the non-periodic theorem, but the blueprint should not suggest that all of its material is a required input for the remaining CPSV comparison. The required source input still missing from the general route is the homogeneous block-injective direct-sum span after blocking,

$$\operatorname{span}\{\tilde{A}^w : |w| = L\} = \bigoplus_{j=1}^{g} M_{D_j}(\mathbb{C}),$$

or its trace-dual separation form. This is the input recorded separately in `docs/paper-gaps/rmp_nonperiodic_bnt_comparison_inputs.tex` and `docs/paper-gaps/pgvwc07_direct_sum_input.tex`.

3 Multiplicative domain versus operator convexity

The Kadison–Schwarz and multiplicative-domain part of the Schwarz chapter is a real dependency of the aperiodic route. It is used to turn peripheral eigenvectors of normalized completely positive maps into multiplicative-domain relations, and then into the commutation and phase-gauge rigidity needed in the spectral and canonical-form chapters. Consequently this material should remain before Perron–Frobenius theory and the non-periodic Fundamental Theorem.

The operator convexity, Jensen, and Lieb-concavity material is different. A label audit finds no current use of that section in the non-periodic Fundamental Theorem chapter or its BNT comparison chain. Its natural downstream uses are entropy, MPDO, and parent-Hamiltonian arguments, especially the open operator-convexity and POVM-resolution work around the parent-Hamiltonian gap. Therefore the safe first step is not to move the whole Schwarz chapter. The safe split, if desired in a later PR, is to keep Kadison–Schwarz and the multiplicative domain upstream, and to move only the operator-convexity/Jensen section to a downstream analytic chapter after the Fundamental Theorem.

4 Which assumptions are extra

In the conditional after-blocking statements, the following data are not extra mathematical assumptions relative to the source theorem once the upstream canonical-form reduction has been proved: a common positive blocking length, zero-tail equality

$$A^{[p]} \sim \mathbf{0}_{d^p, z_A} \oplus A_{\mathrm{nz}}, \qquad B^{[p]} \sim \mathbf{0}_{d^p, z_B} \oplus B_{\mathrm{nz}}, \qquad z_A = z_B,$$

and trace-preserving, primitive, irreducible nonzero sectors. These are the formal names for the canonical-form and period-removal output in the source route.

The remaining hypotheses are genuine proof obligations in the current conditional after-blocking statements. They are one-site injectivity at the chosen blocking length and the BNT block-matching input needed for the representative comparison. The source proves this matching from the proportional MPV hypothesis together with the canonical-form BNT structure; the former formal coefficient-array route, with nonzero limiting coefficients, has been deleted because it is not the source theorem surface. The required block-matching conclusion gives the permutation, matched bond dimensions, gauges, and nonzero phases satisfying

$$B^i_{\pi(j)} = \zeta_j X_j A^i_j X_j^{-1}.$$

The formal statements which assume this block-matching input do not prove it from equality of MPV families. They only transport it through the common blocking and then recover the sector weights. This is why the blueprint and the paper-gap notes must continue to mark the passage as conditional. In particular, this audit does not claim that the proof gaps tracked by issues 1499, 1500, or 1501 have been solved.

5 Staged blueprint organization

The chapter order can be improved without changing theorem statements. A safe organization is to keep the aperiodic proof dependencies first, ending with the proof of the Fundamental Theorem. The direct periodic theorem, symmetries and string order, and the algebraic fixed-length theorem then form the first post-FT block. Entropy and MPDO material should be placed before parent Hamiltonians, since the parent-Hamiltonian chapter depends conceptually on state and density-operator technology rather than conversely.

The larger split of the Schwarz chapter should be staged separately. It is a pure organization change once the operator-convexity section is isolated, but it touches many labels and should not be combined with proof-gap claims.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.

[DlCCSPG17] Gemma De las Cuevas, J. Ignacio Cirac, Norbert Schuch, and David Pérez-García. Irreducible forms of matrix product states: Theory and applications. *Journal of Mathematical Physics*, 58(12):121901, 2017.

[MGRPG+18] András Molnár, José Garre-Rubio, David Pérez-García, Norbert Schuch, and J. Ignacio Cirac. Normal projected entangled pair states generating the same state. *New Journal of Physics*, 20(11):113017, 2018.