

Fixed-Length Chain Equality Does Not Imply Combined-Tensor MPV Equality

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Verdict: unfaithful (resolved).

1 The incompatible statements

For a site-dependent closed chain $\mathbf{A} = (A_0, \dots, A_{n-1})$, equality of the fixed-length states tests only cyclic products containing one matrix from each site, in their cyclic order. In contrast, MPV equality for the tensor obtained by combining the site and physical indices tests every word in the enlarged alphabet. Such a word may repeat one site and omit every other site. Therefore fixed-length state equality cannot imply combined-tensor MPV equality.

This distinction is visible already for physical and bond dimensions one and chain length three. Take

$$A_0 = A_1 = A_2 = 1, \quad B_0 = 2, \quad B_1 = \frac{1}{2}, \quad B_2 = 1.$$

Every local tensor of both chains is injective, and the two closed-chain coefficients agree because $A_0 A_1 A_2 = B_0 B_1 B_2 = 1$. The combined tensors do not generate the same MPV: the length-one word selecting site zero has coefficient 1 for the combined tensor of $\mathbf{A}$ and coefficient 2 for that of $\mathbf{B}$.

Consequently the former abstract same-state reduction assertion has no construction even after adding injectivity of the second chain.¹ It is not the blocking argument in [MGRPG⁺18, Theorem thm:inj_MPS, lines 688–725].

2 The source-faithful route

The source assumes that every local tensor in both chains is injective and derives one invertible gauge per bond directly from equality of the single closed-chain state. This statement is formalized directly in Chapter 24 of the blueprint.² It does not pass through an all-length MPV assertion for the combined tensor.

¹Former formal declaration: `MPSChainTensor.SameStateBridgeHyp`, previously in `TNLean/MPS/Chain/SameStateBridge.lean`.

²Formal declaration: `TNLean.PEPS.fundamentalTheorem_injectiveMPSChain_of_sameState` in `TNLean/PEPS/CycleMPSChainOverlapCapstone`.

For constant chains, equality at one length likewise determines a tensor only up to a scalar whose n -th power is one and a similarity transformation. A phase-free conclusion from one fixed length is false: in bond dimension one, replace $A = 1$ by $B = \omega$ for a nontrivial n -th root of unity ω . The length- n states agree, but scalar conjugation cannot change 1 into ω . The all-length hypothesis in the corresponding translation-invariant reduction rules out this phase and remains a separate, stronger theorem.³

3 Verdict

Classification: unfaithful theorem. The abstract same-state reduction hypothesis asserted a mathematically false implication not supplied by the cited source, so it and its consumers are removed. The fixed-length injective-chain Fundamental Theorem is represented only by the direct source-faithful theorem in Chapter 24. The Chapter 23 results based on combined-tensor MPV equality remain explicitly all-length results and are not presented as consequences of fixed-length state equality.

References

- [MGRPG⁺18] András Molnár, José Garre-Rubio, David Pérez-García, Norbert Schuch, and J. Ignacio Cirac. Normal projected entangled pair states generating the same state. *New Journal of Physics*, 20(11):113017, 2018.

³Formal declaration: `MPSChainTensor.ti_reduction_corollary` in `TNLean/MPS/Chain/TranslationInvariance`.