

One-Dimensional Scope of Schumacher–Werner Support-Algebra Covariance

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Verdict: scope-restriction (open).

1 The source identity

Schumacher–Werner define a quantum cellular automaton on the cube lattice $\mathbb{Z}^s$ with a finite neighbourhood scheme [SW04, lines 309–318]. For their main structure theorem they specialize to the cubic nearest-neighbour scheme, group the 2^s sites of the unit cube $\square$, and let $Q = \{-1, +1\}^s$ denote the quadrant vectors [SW04, lines 920–941]. For each $q \in Q$, they define the support algebra

$$\mathcal{B}_q = \text{Spp}(\gamma(\mathcal{A}(\square)), \mathcal{A}(\square + q)). \quad (1)$$

Here γ is the translation-commuting algebra homomorphism of their QCA definition; invertibility is not assumed. Their translated support calculation states that

$$\text{Spp}(\gamma(\mathcal{A}(\square + q_1 - q)), \mathcal{A}(\square + q_1)) = \tau_{q_1 - q}(\mathcal{B}_q). \quad (2)$$

The definition and calculation occur at lines 1199–1206 and 1218–1226 of Ref. [SW04]. They apply in arbitrary lattice dimension with this cubic nearest-neighbour scheme.

2 The formal specialization

The present quasi-local observable algebra is a homogeneous chain on $\mathbb{Z}$ with one fixed matrix size $d > 0$. For a star-automorphism ω satisfying

$$\omega(\mathcal{A}_\Lambda) \subseteq \mathcal{A}_{\Lambda + \{-1, 0, 1\}}, \quad (3)$$

$$\omega\tau_a = \tau_a\omega \quad (a \in \mathbb{Z}), \quad (4)$$

the support algebras are formed from the even source pair $E_x = \{2x, 2x + 1\}$ and its two neighbouring target pairs. Translation by two sites preserves this pairing and gives

$$\tau_2(\widehat{\mathcal{R}}_y) = \widehat{\mathcal{R}}_{y+2}. \quad (5)$$

This is the one-dimensional, nearest-neighbour, homogeneous-chain instance of (2). The fixed on-site dimension agrees with the QCA setting used by Cirac et al. at lines 2292–2298 of Ref. [CPGSV17]; the ordered even and odd support factors agree with equation RR2x of Ref. [GNVW12].

The formal statement assumes the propagation and translation covariance in (3)–(4), but it packages ω as a star-automorphism. Schumacher–Werner require only a translation-commuting algebra homomorphism. Thus the formal theorem is additionally restricted to the invertible case and does not formalize the arbitrary-dimensional cube lattice.¹ The separate restriction from site-dependent GNVW cell sizes to a homogeneous chain is recorded in `docs/paper-gaps/gnvw12_site_dependent_adjacent_generation_scope.tex`.

3 Elimination plan

Removing the invertibility restriction first requires homomorphism-valued versions of finite propagation, finite local restriction, translation covariance, and the site-indexed support-algebra family. The compression proof uses multiplication, addition, the star operation, and translation commutation; it does not require an inverse for ω .

Removing the dimensional restriction then requires a quasi-local observable algebra indexed by $\mathbb{Z}^s$, finite-region translations by vectors in $\mathbb{Z}^s$, and a propagation predicate for the cubic nearest-neighbour scheme. For each translated source region and each target cube, the support algebra must then be embedded canonically into the multidimensional quasi-local algebra. Translation covariance and the same coefficient-compression argument yield the vector-valued analogue of (5), recovering (2). The present theorem follows by specializing to $s = 1$, $\mathcal{N} = \{-1, 0, 1\}$, and the two-site blocked pairing.

Until both interfaces are available, the source-labelled formal statement remains restricted to star-automorphisms of the one-dimensional nearest-neighbour chain described above.

References

- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product unitaries: Structure, symmetries, and topological invariants. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(8):083105, 2017.
- [GNVW12] David Gross, Vincent Nesme, Héctor Vogts, and Reinhard F. Werner. Index theory of one dimensional quantum walks and cellular automata. *Communications in Mathematical Physics*, 310(2):419–454, 2012.

¹Formal declaration: `SpinChain.PropagatesWithin.embeddedSupportAlgebra_map_quasiLocalTranslation_two` in [TNLean/QCA/SupportAlgebraCovariance](#).

- [SW04] Benjamin Schumacher and Reinhard F. Werner. Reversible quantum cellular automata. Preprint, 2004.