

Strong and Weak Symmetries of Matrix Product Density Operators: Arbitrary Commuting Boundaries

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Verdict: scope-restriction (resolved).

1 The source definitions

Sun considers translation-invariant density operators

$$\rho^{(L)}(\mathcal{X}, M) = \sum_{i,j} \text{tr}[\mathcal{X} M^{i_1 j_1} \dots M^{i_L j_L}] |i_1 \dots i_L\rangle \langle j_1 \dots j_L|, \quad (1)$$

where the boundary matrix $\mathcal{X}$ commutes with every M^{ij} , and calls such a family strongly symmetric under matrix product operators $O_a^{(L)}$ when $O_a^{(L)} \rho^{(L)} = \lambda_a^{(L)} \rho^{(L)}$ for all a and L , and weakly symmetric when $[O_a^{(L)}, \rho^{(L)}] = 0$ for all a and L [Sun25].¹ Every purification Ψ of a strongly symmetric $\rho^{(L)}$ satisfies $(O_a^{(L)} \otimes \mathbb{I}_{\text{anc}}) \Psi = \lambda_a^{(L)} \Psi$.²

2 What is formalized

The operator family (1) is formalized for an arbitrary virtual boundary $\mathcal{X}$. The boundary matrices commuting with all letters form their centralizer, a complex subalgebra. Positivity of the resulting operators at every positive length is a separate condition: commutation alone does not imply positivity.

The family predicates for strong and weak symmetry retain the boundary $\mathcal{X}$. The eigenvalue $\lambda_a^{(L)}$ may depend on L and need not be a phase. Strong symmetry implies weak symmetry for a positive family under the source’s proviso that the adjoint of every operator belongs to the family. Fusion and on-site character relations apply to the boundary-weighted family wherever the operator is nonzero.

¹Source traceability: `References/2504.16985/main.tex`, lines 175–182.

²Source traceability: lines 183–187.

For any supplied purification $\rho^{(L)}(\mathcal{X}, M) = \Phi_L \Phi_L^\dagger$, strong symmetry is equivalent to $O_a^{(L)} \Phi_L = \lambda_a^{(L)} \Phi_L$. This is the source’s purification lemma for arbitrary commuting boundaries, and requires no factorization of $\mathcal{X}$ on the virtual space. The vectorized-state identities likewise retain $\mathcal{X}$ in the word traces and prove the equivalences with the actions $U_g \otimes \overline{U_g}$ and $U_g \otimes \mathbb{K}$. The identity-boundary statements follow by setting $\mathcal{X} = \mathbb{K}$.

3 Local purification statements

A commuting boundary on a doubled virtual space need not factor as $Y \otimes \overline{Y}$, nor does it automatically define a positive family. The existing local-purifying-tensor sufficient conditions and the converse characterization using a normal periodic purification therefore remain explicit identity-boundary results. They are not arbitrary-boundary local gauge characterizations. This does not restrict the family definitions, the general global-purification lemma, or the boundary-weighted vectorization identities.

4 Resolution

The identity-boundary restriction of the source’s family predicates and purification lemma is removed. The corresponding declarations are `MPOTensor.mpowithBoundary`, `MPOTensor.commutingBoundaryAlgebra`, `MPOTensor.IsMPDOWithBoundary`, `MPOTensor.IsStrongMPOSymmetry`, `MPOTensor.IsWeakMPOSymmetry`, and `MPOTensor.isStrongMPOSymmetry_iff_purification`.

References

- [Sun25] Xiao-Qi Sun. Anomalous matrix product operator symmetries and 1D mixed-state phases. Preprint, 2025.