

Bond Dimensions of the Successive Decompositions and Minimal Resources

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Verdict: scope-restriction (open).

1 Source range

After the successive singular value decompositions that write a state as $|\tilde{\psi}\rangle = V'_{[n]} \cdots V'_{[1]} |\varphi_I\rangle$, Ref. [SSV⁺05, lines 200–202 of the source] states: “Simple rank considerations show that $V'_{[n-k]}$ has dimension $2 \min[D, 2^k] \times \min[D, 2^{k+1}]$.” Ref. [PGVWC07, Section 5.1, lines 1574–1577] adds that this proof “also provides a *recipe* for the generation of any given state (with minimal resources).” Ref. [PGVWC07, lines 431–458] states that every state of N sites has an open-boundary representation with bond dimension $D \leq d^{\lfloor N/2 \rfloor}$ satisfying three gauge conditions, and that “any state for which $\max_m \text{rank}(\rho_m) \leq D$ can be written as a MPS of bond dimension D ”.

2 What is formalized

For local dimension d , the isometric open-boundary representation produced by the decompositions of a given representation has bond dimension at most $\min(D, d^k)$ at the bond after k sites counted from the end where the decompositions start, and, after running the decompositions once from each end, at most $\min(D_k, d^k, d^{N-k})$, where $D_k \leq D$ is the bond dimension of the given representation at that cut. Read as a statement that isometries of the printed size suffice, the first quoted sentence is formalized with the source’s hypotheses.

For the cut k , let $\text{rank}_k \psi$ be the rank of the matrix $(\sigma, \tau) \mapsto \psi(\sigma\tau)$ whose rows are the configurations of the first k sites and whose columns are those of the remaining $N - k$ sites; it is the rank of the reduced density operator ρ_k . The following are formalized on chains of $N \geq 1$ sites.

1. Every open-boundary representation of ψ has bond dimension at least $\text{rank}_k \psi$ at the cut k .
2. If $\psi \neq 0$ and $\text{rank}_k \psi \leq D$ for every k , successive Schmidt decompositions of ψ give an open-boundary representation with common bound D whose

bond dimension at the cut k equals $\text{rank}_k \psi$ for every k , and in which every site except the last satisfies $\sum_i A_i^\dagger A_i = \mathbb{I}$ (every site, when ψ is normalized).

3. Consequently the states generated with a D -dimensional ancilla, probabilistically or deterministically, are those whose cut ranks are all at most D (normalized, in the deterministic case), and the least ancilla dimension that generates ψ is $\max_k \text{rank}_k \psi$. This is the “minimal resources” of the second quoted sentence.
4. Every state of $N \geq 1$ sites has an open-boundary representation with bond dimension at most $d^{\lfloor N/2 \rfloor}$, since $\text{rank}_k \psi \leq \min(d^k, d^{N-k})$.

3 Readings that are not formalized

The printed size $2 \min[D, 2^k] \times \min[D, 2^{k+1}]$ is not the minimal size for low-rank states: a product state has $\text{rank}_k \psi = 1$ at every cut, so a representation with all bond dimensions one exists, whereas $\min[D, 2^k] \geq 2$ for $D \geq 2$ and $k \geq 1$. A representation of the printed size still exists, since unused virtual directions can be added to a smaller one, so the sentence is not refuted when read as asserting that isometries of exactly that size can be chosen; it is only not minimal. Only the upper-bound reading is formalized, and the minimal sizes are the cut ranks of the previous section.

A single pass of the decompositions over a given representation need not reach the cut ranks: if a bond direction of the given representation is annihilated by the sites to its right, the left-to-right pass can keep it. The formalization obtains the minimal bond dimensions from the Schmidt decompositions of ψ itself, as in Ref. [PGVWC07, lines 444–447].

The gauge conditions 1–3 of the completeness theorem [PGVWC07, lines 431–442], namely $\sum_i A_i^{[m]} A_i^{[m]\dagger} = \mathbb{I}$, the relation between consecutive diagonal matrices $\Lambda^{[m]}$, and the positivity and normalization of $\Lambda^{[m]}$, are not formalized; only the existence statement with the bound $D \leq d^{\lfloor N/2 \rfloor}$ is.

4 Elimination plan

The restrictions above concern the minimality of the printed size and the canonical gauge of the completeness theorem. The latter is formalized by diagonalizing the reduced density operators ρ_m in the representation of item 2 and rescaling the bond bases by the square roots of their eigenvalues.

References

- [PGVWC07] D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac. Matrix product state representations. *Quantum Information & Computation*, 7(5):401–430, 2007.

- [SSV⁺05] C. Schön, E. Solano, F. Verstraete, J. I. Cirac, and M. M. Wolf. Sequential generation of entangled multiqubit states. *Phys. Rev. Lett.*, 95:110503, 2005.