

Uniform Parent-Hamiltonian Gaps for Compact Normal Families

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Verdict: scope-restriction (resolved).

1 What the source states

Appendix A of [SPGC11] proves that the family H_γ of parent Hamiltonians along the path from a matrix product state to its isometric form has a spectral gap bounded below uniformly in $\gamma \in [0, 1]$ and in the chain length. The tensors of the path are in normal form, $A_i(\gamma) = \bigoplus_\alpha A_i^\alpha(\gamma)$, possibly with several blocks. The argument applies Nachtergaele’s bound, controlled by the second transfer eigenvalue of each block, by the overlaps of the local spaces of different blocks, and by the gap $\Delta_{2p}(\gamma)$ of H_γ restricted to $2p$ sites. The last quantity has a uniform lower bound “as the restricted Hamiltonian is continuous in γ ”.¹

2 The earlier restriction

The first compact-family result applied only to one-site injective tensors and the canonical two-site interaction. It did not cover normal forms with several blocks, or the positive non-projector interactions of the source. That restriction is removed by the results below. The proof uses finite-range Knabe windows in place of the source’s Nachtergaele estimate; this changes the argument, not the uniform-gap conclusion. Compact families of normal forms other than the source path are covered below only under a common simultaneous spanning length or at the stated sufficient ranges; this is not part of the source’s statement.

3 Compact normal-form families

Let $x \mapsto (A_x^j)_j$ be continuous, with fixed positive block dimensions D_j , and suppose the length- S word tuples span the product matrix algebra for every parameter, where $S > 0$. Let A_x be their block sum with nonzero weights. The

¹Source traceability: `Papers/1010.3732/paper_v3.tex`, lines 2475–2580; the continuity step is lines 2575–2578.

weights need not be continuous, since they do not affect any local ground space. For every fixed $R \geq S + 1$ and every compact parameter set K , there is $\delta > 0$ such that

$$\delta \|v\| \leq \|H_{N,R}(A_x)v\| \quad (x \in K, N \geq R, v \perp \ker H_{N,R}(A_x)).$$

This includes the minimal ring $N = R$. The joint boundary map identifies the local ground space, while its restriction to scalar block boundaries identifies the periodic kernel. Both maps are injective and continuous, so their range projections are continuous. Compactness supplies uniform gaps for finitely many short rings. Strict finite-range Knabe windows and a finite subcover supply the common gap for all longer rings.

For pairwise inequivalent normalized normal blocks, a common simultaneous spanning length is derived from the dimension bound. Any $R \geq 3 \max(\sum_j D_j, 1)^5$ suffices. For a single normal tensor, without a normalization hypothesis, quantum Wielandt gives the sharper sufficient range $D^4 + 1$. These bounds are sufficient, not asserted to be optimal.²

4 Positive interactions and the source path

For a continuous family of positive local interactions h_x with the same local kernels, there is $\kappa > 0$, uniform on a compact parameter set, such that $h_x \geq \kappa P_{G_R(A_x)}^\perp$. Summing translated copies transfers the canonical gap to these interactions. A pointwise upper comparison gives equality of the periodic kernels; its constant need not be uniform.³

The source first blocks the tensor so that the joint one-site boundary map is injective. Along its isometric deformation, this map is multiplied on the physical space by the invertible map Q_γ . Simultaneous spanning therefore persists with $S = 1$. The preceding result applies at $R = 2$ to the source's continuous positive interactions. Thus the absence of several blocks and non-projector interactions in the earlier result is no longer a restriction.

The particular isometric deformation is also constructed from the original block tensor. Let $P_{i,(j,a,b)} = A_i^j(a,b)$ and let E be the orthogonal projection onto $\text{ran } P$. The left polar factors satisfy $P = QW$, $Q = (PP^\dagger)^{1/2} + I - E > 0$, and $WW^\dagger = E$. The identity extension outside the retained physical space leaves the factorization unchanged and keeps the original physical dimension. Reading W as block matrices gives a continuous path $Q_\gamma W$ whose endpoints are W and the original tensor. Its simultaneous word spans are those of the original blocks, and both its canonical projections and the source's positive interactions

²Formal declarations: `MPSTensor.exists_uniform_parentHamiltonianES_toTensorFromBlocks_gap_of_compact_all_lengths`, `MPSTensor.exists_uniform_block_parentHamiltonianES_gap_of_compact_isNormalTensor`, and `MPSTensor.exists_uniform_parentHamiltonianES_gap_of_compact_isNormal_all_lengths`.

³Formal declarations: `MPSTensor.exists_uniform_pos_parentInteractionES_le_of_compact`, `MPSTensor.exists_uniform_periodicInteractionHamiltonianES_gap_of_compact_canonical_gap`, and `MPSTensor.exists_uniform_block_parentInteraction_gap_of_compact`.

have a uniform periodic gap for all $N \geq 2$. For pairwise inequivalent normal blocks, the required initial blocking is derived at a positive length L satisfying $L + 1 \leq 3 \max(\sum_j D_j, 1)^5$.⁴

For a one-site injective tensor, the deformation is also formalized through the right polar factors. Writing its physical matrix as $M = VP$, the path $M_\gamma = V((1-\gamma)I + \gamma P)$ is continuous and injective in the original physical space, even when M is rectangular. The positive factor stays invertible throughout the interval. Unitary physical and virtual covariance is preserved, and the canonical parent interactions retain the on-site symmetry. This covers only the single-block case of Section II.C.⁵

The canonical parent projections along any continuous one-site injective path with fixed on-site symmetry now also give a symmetric gapped interaction path on the common physical space. The norm-gap estimate gives a spectral gap above zero, since the periodic MPS vector is a nonzero zero eigenvector. Applied to the polar path in unitary virtual gauge, this joins the canonical parents of the original tensor and its isometric form. It does not perform the different-dimension endpoint embeddings in the classification theorem.⁶

Normalization and unitary virtual covariance are now obtained from the original one-site injective on-site symmetric tensor. Perron normalization uses only nonzero rescaling and virtual gauge, which preserve all canonical parent interactions. The positive definite adjoint fixed point is derived from unitality by trace duality, and the virtual gauges are made unitary. Thus the path to the isometric parent requires neither an assumed normalization nor supplied virtual unitary matrices.⁷

The prepared path now also retains the virtual cohomology class of the original tensor. A bond change of basis conjugates all virtual matrices without changing their factor system. Unitary covariance choices for the prepared tensor assemble into a single projective representation, which implements symmetry at every point of the polar path. Gauge independence then identifies the class with any endpoint choice.⁸

5 General compact joint boundary inputs

The joint boundary map is injective exactly when the homogeneous word tuples span the product of the block matrix algebras. At a positive initial length, this

⁴Formal declarations: `Matrix.leftPolarPos_mul_leftPolarIso`, `Matrix.range_leftPolarIso_finalProjection`, `MPSTensor.exists_uniform_isometricDeformation_parent_gap`, `MPSTensor.exists_uniform_isometricDeformation_interaction_gap`, and `MPSTensor.exists_blocked_isometricDeformation_parent_gap_of_isNormalTensor`.

⁵Formal declaration: `MPSTensor.exists_uniform_parentHamiltonianES_gap_polarDeformation`.

⁶Affected declarations: `MPSTensor.canonicalInjectiveGappedPath` and `MPSTensor.polarGappedInteractionPath`.

⁷Affected declaration: `MPSTensor.exists_prepared_polarGappedInteractionPath_of_isOnSiteSymmetric`.

⁸Affected declaration: `MPSTensor.exists_prepared_polarGappedInteractionPath_with_virtual_class`.

property persists at every larger length without normalization or nonzero block-dimension assumptions. Openness and compactness therefore give one positive threshold for a compact continuous family with pointwise positive-length spanning. Trace preservation also allows an initial length of zero by adjoining one letter. For normalized primitive blocks that are pairwise inequivalent under similarity and phase, the pointwise hypothesis follows without requiring continuous fixed-point matrices.

These continuity results supplement the gap conclusions above. The compact normal-form gap statements remain unchanged. The weighted block family's local ground projection, canonical interaction, and fixed-volume Hamiltonians are continuous at a common injectivity length. Nonzero block weights need not vary continuously.⁹

References

- [SPGC11] Norbert Schuch, David Pérez-García, and Ignacio Cirac. Classifying quantum phases using matrix product states and projected entangled pair states. *Physical Review B*, 84(16):165139, 2011.

⁹Formal declarations: `MPSTensor.exists_uniform_wordTupleSpanTop_of_compact`, `MPSTensor.wordTupleSpanTop_of_ge_of_pos`, `MPSTensor.exists_uniform_blockGroundSpaceMapES_injective_of_compact_isPrimitiveMPS`, `MPSTensor.continuous_iSup_groundSpaceES_starProjection_of_injective_family`, `MPSTensor.continuous_openParentHamiltonianES_toTensorFromBlocks_family`, and `MPSTensor.continuous_parentHamiltonianES_toTensorFromBlocks_family`.