

The Virtual Unitary in the Symmetric Isometric Deformation

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Verdict: scope-restriction (resolved).

1 The source assertion

Schuch, Pérez-García, and Cirac [SPGC11] show that the isometric deformation can preserve an imposed on-site symmetry. The local source is `Papers/1010.3732/paper_v3.tex`, lines 645–676. Their argument first puts the tensor map P in a standard virtual gauge, then associates to a physical symmetry U a virtual unitary X with $UP = PX$. Writing $P = QW$, they deduce $[U, Q] = 0$ and preservation of the symmetry by $P_\gamma = Q_\gamma W$ and its parent Hamiltonian.

2 The proved polar implication

For any rectangular matrix P and supplied unitaries U, X satisfying $UP = PX$, one has $[U, PP^*] = 0$. Let E project onto $\text{ran } P$. The positive factor used on the original physical space is

$$\begin{aligned} Q &= (PP^*)^{1/2} + I - E, \\ P &= QW. \end{aligned} \tag{1}$$

Functional calculus gives $[U, (PP^*)^{1/2}] = [U, E] = 0$, hence $[U, Q] = 0$. Since $Q > 0$, cancellation in $UP = PX$ gives $UW = WX$. Thus $Q_\gamma = \gamma Q + (1 - \gamma)I$ commutes with U , and $UQ_\gamma W = Q_\gamma WX$. These statements are proved in `TNLean/MPS/ParentHamiltonian/IsometricDeformationCovariance.lean`. The identity extension in (1) is included: its square is not identified with PP^* outside $\text{ran } P$.

3 The parent-space implication

Suppose in addition that the original two-site MPS boundary space is invariant under $U \otimes U$. Invertibility of Q_γ and commutation with U preserve this invariance along the deformation. The orthogonal parent projection then commutes with $U \otimes U$. The source’s positive interaction

$$h_\gamma = (Q_\gamma^{-1})^{\otimes 2} h_0 (Q_\gamma^{-1})^{\otimes 2} \tag{3}$$

has the same commutation property. Each embedded local interaction commutes with $U^{\otimes N}$, so both open and periodic interaction sums preserve the symmetry. These conditional statements are proved in `TNLean/MPS/ParentHamiltonian/IsometricDeformationSymmetry.lean`.

This implication requires invariance of the original MPS boundary space. An arbitrary virtual unitary X satisfying $UP = PX$ need not preserve the virtual contractions defining an MPS. Thus the rectangular covariance identity by itself is not substituted for the original MPS symmetry.

4 The proved single-block case

For a single already-injective block A , assume either the trace-preserving canonical normalization $\sum_i A_i^* A_i = I$ or its unital counterpart $\sum_i A_i A_i^* = I$, and exact invariance of its periodic matrix product vectors under a

unitary on-site representation. Then the fundamental theorem supplies a virtual similarity, and uniqueness of the canonical fixed point makes that similarity unitary up to a scalar. Conjugation by the resulting virtual unitary gives a unitary action on matrix coordinates. This derives $UP = PX$ and $[U, Q] = 0$ without supplying a virtual unitary as an additional hypothesis. State symmetry also gives invariance of the original two-site boundary space. Hence both canonical parent projections and the positive interactions in (3) preserve the symmetry throughout the deformation. The derivation is proved in `TNLean/MPS/ParentHamiltonian/InjectiveIsometricDeformationSymmetry.lean`, and its parent-Hamiltonian consequences in the preceding symmetry module.

The normalization convention requires a coordinate qualification. For $P_{i,(a,b)} = A_i(a,b)$, the source's $\text{tr}_{\text{left}}(P^*P) = I_{\text{right}}$ at line 653 is $\sum_i A_i^* A_i = I$. The two proved cases are stated separately in their respective canonical orientations; the trace-preserving case matches this source convention. Both assume exact vector symmetry; invariance up to a phase is treated below for one or several blocks in the trace-preserving convention.

5 The several-block case

Let $A^1, \dots, A^r$ be blocks of positive bond dimensions whose one-site tuples $(A_i^k)_k$ span $\bigoplus_k M_{D_k}$; this joint span is the source's standard form (lines 325–352, and in matrix language lines 1740–1743). Suppose first that $U_g^{\otimes N}$ fixes every matrix product vector of $\bigoplus_k A^k$. Equality of all word traces of the original and twisted block families defines an algebra automorphism of $\bigoplus_k M_{D_k}$ that sends each one-site tuple of the original blocks to the twisted one. Such an automorphism permutes the blocks and is inner on each block. Hence the twist of block $\sigma(k)$ is gauge equivalent to block k for a permutation σ preserving the bond dimensions. If every block satisfies $\sum_i A_i^{k\dagger} A_i^k = I$, which is the printed condition $\text{tr}_{\text{left}}(P^\dagger P) = I$ for the block-diagonal P , each gauge may be taken unitary. The resulting joint virtual action, a block permutation composed with blockwise unitary conjugations, is unitary and satisfies $U_g P = P X_g$. It therefore gives $[U_g, Q] = 0$. Gauge equivalence of the permuted blocks also gives invariance of every local MPS space of $\bigoplus_k A^k$. Both supplied hypotheses of the conditional parent-space implication are thereby derived, and the parent-symmetry conclusions hold along the whole deformation. These statements are proved in `TNLean/MPS/ParentHamiltonian/BlockIsometricDeformationSymmetry.lean`.

6 Invariance up to a phase

The source's symmetry is a symmetry of the state, which is invariant under U_g only up to a phase (lines 1313–1314 and 2590–2593). The faithful hypothesis is therefore $U_g^{\otimes N} |\psi_N\rangle = \lambda_{g,N} |\psi_N\rangle$ for every N , with scalars $\lambda_{g,N}$ that may a priori depend on both g and N ; no character property of λ is assumed.

Write $\phi(M) = \sum_k \text{tr } M_k$ on $\bigoplus_k M_{D_k}$, let $x_i = (A_i^k)_k$ and let $y_i = (\sum_l (U_g)_{il} A_l^k)_k$. Both families span the direct sum, the pairing $(M, N) \mapsto \phi(MN)$ is nondegenerate, and $\phi(1) = \sum_k D_k \neq 0$. The hypothesis at lengths one, two and three says $\phi(y_w) = c_{|w|} \phi(x_w)$ for words w of those lengths. Nondegeneracy against the spanning family y makes the assignment $x_i \mapsto y_i$ a well-defined linear map L , and then

$$\phi(LX) = c_1 \phi(X), \quad (4)$$

$$\phi(LX LY) = c_2 \phi(XY), \quad (5)$$

$$\phi(LX LY LZ) = c_3 \phi(XYZ) \quad (6)$$

for all X, Y, Z . Pairing against LZ , which ranges over the whole algebra, gives $c_1 L(1) = c_2 I$ and $LX LY = L(XY) L(1)$. Since L is onto, $c_1 \neq 0$, and comparing $\phi(L(1)) = c_1 \phi(1)$ gives $c_2 = c_1^2$ and $L(1) = c_1 I$. Hence $LX LY = c_1 L(XY)$, and induction on the word length gives $\phi(y_w) = c_1^{|w|} \phi(x_w)$ for every word. At length one the vector $v_i = \phi(x_i)$ is nonzero and satisfies $U_g v = c_1 v$, so unitarity of U_g gives $|c_1| = 1$. Consequently $\lambda_{g,N} = c_1^N$ whenever $|\psi_N\rangle \neq 0$.

The rescaled unitary $\bar{c}_1 U_g$ preserves every summed word trace, so the several-block derivation above applies to it. The twist of block $\sigma(k)$ by U_g is c_1 times a unitary conjugate of block k , the virtual action $c_1 X_g$ is unitary and satisfies $U_g P = P(c_1 X_g)$, and $[U_g, Q] = 0$. Scaling $U_g^{\otimes L}$ by the nonzero scalar c_1^L does not change the image of a subspace, so every local MPS space is invariant under $U_g^{\otimes L}$ itself. All parent-symmetry

conclusions of the several-block case therefore hold under symmetry up to a phase. These statements are proved in `TNLean/MPS/ParentHamiltonian/BlockIsometricDeformationSymmetry.lean`; the one-block case $r = 1$ is the already-blocked single injective block in the printed normalization.

7 Status

The source assertion at lines 645–676 is formalized under the source’s hypotheses: a block-diagonal tensor in the printed standard form, whose blocks jointly span at one site, and an on-site unitary representation leaving every matrix product vector invariant up to a phase. The single-block statements in the unital canonical orientation remain stated for exact symmetry; they are not needed for the source assertion, whose printed normalization is covered by the several-block statements with one block. The separately established uniform-gap results for the constructed deformation do not depend on this note.

References

- [SPGC11] Norbert Schuch, David Pérez-García, and Ignacio Cirac. Classifying quantum phases using matrix product states and projected entangled pair states. *Physical Review B*, 84(16):165139, 2011.