

Two-by-two regular PEPS renormalization with the original tensor

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Verdict: scope-restriction (open).

1 Source and exact result

The source is Schuch, Cirac, and Pérez-García, arXiv:1001.3807, Observations 6.5–6.6, local source lines 1818–1915 and the three diagrams `figs4/renorm-g-g-sym.pdf`, `figs4/renorm-g-id-and-split.pdf`, and `figs4/renorm-fixed-point.pdf`.

Let G be any finite group, not necessarily abelian, and let A be a four-leg regular G -isometric tensor with any finite physical alphabet P . On an untwisted doubled torus with coarse periods $w, h \geq 3$, let Ψ_f be the actual fine state and Ψ_c the actual coarse state defined by the same original tensor A . Both states are proved nonzero. There is an isometry on the derived fine physical support, given by physical four-site grouping, a product of local physical maps, and regrouping of output registers, such that

$$I\Psi_f = R\Psi_c \otimes \Omega, \tag{1}$$

$$I\left(\frac{\Psi_f}{\|\Psi_f\|}\right) = \frac{\Psi_c}{\|\Psi_c\|} \otimes \Omega, \quad \|\Omega\| = 1. \tag{2}$$

The exact positive scalar is

$$R = \left(\prod_{v \in \mathbb{Z}_w \times \mathbb{Z}_h} \frac{\sqrt{c_{B,v}}}{\sqrt{c_{C,v}}} \right) (\sqrt{|G|})^{|E(\Gamma)|}. \tag{3}$$

Here $c_{B,v} > 0$ is the actual blocked tensor's isometry factor and $c_{C,v} > 0$ is the isometry factor of the original coarse tensor with its surplus identity registers. All factors are derived from the local G -isometry assumptions. The four-site Gram calculation separately gives $c_{B,v} = |G| \prod_{i=0}^3 c_{v,i}$ for the fine-site factors used in that calculation; the group-order factor records the internal cycle.

2 Geometric and physical content

The fine torus is bijectively tiled by four-site blocks. An explicit bijection identifies every fine horizontal and vertical bond with four internal bonds and four paired crossing bonds per block. The eight separate boundary labels are retained as the two labels on each coarse leg. This derives the global fine-to-coarse contraction identity, without a supplied factorization, Gram matrix, support condition, or nonzero-state hypothesis.

The target is not merely a canonical tensor. Its local coefficient is the original tensor on the reference labels times the equality indicator for all relative surplus labels. Its G-isometry follows from the original local Gram identity, and its graph contraction factors by an explicit edge-label bijection. Thus the original-tensor comparison is a derived physical operation. A separate theorem also gives an isometric equivalence between the original tensor's full physical support and the canonical averaging tensor's physical support.

All four original physical registers of each block are retained. The physical maps act on the actual product of block ranges, pulled back to the fine physical space by the full grouping unitary. The actual fine state is proved to lie in this support. Unused ambient physical directions are permitted, and neither ambient surjectivity nor an unspecified extension to a unitary on those unused directions is assumed.

3 Exact remaining scope

The geometric bond-indexed reblocking is proved for every pair of positive coarse periods, including one and two. The global Bell separation and original-tensor normalized comparison use a simple coarse graph, so they require both coarse periods at least three. Extending that physical support argument to a bond-indexed graph with parallel bonds and loops is still required for the smaller periods; no simple-graph theorem is applied to them here.

The theorem recorded here has identity bonds. Its companion now proves the corresponding uniform map for all native closures and coherent sums; see `scp10_twisted_two_by_two_regular_fixed_point.tex` and `TNLean.PEPS.exists_regularTwoByTwoTwistedTensorIsometry`. The common physical alphabet may be arbitrary, and fine tensors may vary in the geometric and canonical-coarse results. The normalized same-original-tensor statement assumes one homogeneous four-leg tensor, as in the source's fixed-point assertion. No claim about non-regular representations, approximate schemes, or unrestricted boundary conditions is made. These limits keep the unrestricted source claim separate from this proved periodic specialization.

4 Declarations

The exact geometric identity is `TNLean.PEPS.torusBondNetwork_eq_twoByTwoBlocked`. The derived four-site isometry is `TNLean.PEPS.isGIsometric_t`

`woByTwoSeparatedTensor`. The canonical-coarse global support isometry is `TNLean.PEPS.exists_regularTwoByTwoPhysicalSupportIsometry`. The final original-tensor, scalar-correct and normalized statement is `TNLean.PEPS.exists_regularTwoByTwoOriginalTensorIsometry`. Nonvanishing is derived by the original identity closure sector, and the factorization of the augmented original tensor is `TNLean.PEPS.graphBondNetwork_regularGraphSurplusSite`.