

Commuting Parent Hamiltonians of G -Isometric Tensors: the Projector and the Scope

2026-10-05

Verdict: scope-restriction (open).

1 What the source prints

Section 6.4 of [SCPG10] shows that the parent Hamiltonians of G -isometric PEPS are sums of commuting local terms, and announces that the proof is given for matrix product states, the generalization to PEPS being straightforward.¹ For a G -isometric matrix product tensor A , on whose bond the left-regular representation acts, let

$$\mathcal{S}_2 = \left\{ \sum_{ij} \text{tr}[A^i A^j X] |ij\rangle \mid X \right\}. \quad (1)$$

The lemma of that section states that the local terms h_i of the parent Hamiltonian, “which project onto the complement of $\mathcal{S}_2$ ”, are of the form of equation “eq:iso:ham-proj-from-A”, the two-site operator

$$P = \sum_{ijkl} \text{tr}[A^i A^j (A^k A^l)^\dagger] |ij\rangle \langle kl| \quad (2)$$

drawn in figure “ham-proj-from-A”. Its proof shows that P is a projector, that its range lies in $\mathcal{S}_2$, that it fixes every vector of $\mathcal{S}_2$, and that it is self-adjoint (figures “ham-proj-from-A-is-proj” and “ham-proj-from-A-pres-subspace”). The theorem that follows states that for G -isometric PEPS the terms h_i of the parent Hamiltonian of the two-dimensional parent-Hamiltonian theorem commute. Its proof writes the products of the operators P on sites 1, 2 and 2, 3 as the two sides of figure “ham-comm-step1”, which are equal because the sum over g of the virtual operators $U_g \otimes U_g^{-1}$ can be moved from one bond to the other, and closes with “the same proof applies to two dimensions”.²

¹Source traceability: Papers/1001.3807/paper_v3.tex, lines 2092–2096.

²Source traceability: Papers/1001.3807/paper_v3.tex, lines 2098–2153, figures Papers/1001.3807/figs4/ham-proj-from-A.pdf, Papers/1001.3807/figs4/ham-proj-from-A-is-proj.pdf, Papers/1001.3807/figs4/ham-proj-from-A-pres-subspace.pdf, Papers/1001.3807/figs4/ham-comm-step1.pdf and Papers/1001.3807/figs4/ham-comm-step2.pdf.

2 Local fix: the operator is the projector onto $\mathcal{S}_2$

The local term of the parent Hamiltonian is $h = 1 - \Pi_{\mathcal{S}}$.³ The proof of the lemma shows that P is the orthogonal projector onto $\mathcal{S}_2$, not onto its complement, so the operator of equation “eq:iso:ham-proj-from-A” is $1 - h_i$. The commutation theorem is unaffected: h_1 and h_2 commute exactly when $1 - h_1$ and $1 - h_2$ do. In addition the source omits normalizations in diagrams. For a tensor A with matrices A^i on the bond space $\mathbb{C}[G]$, let $\mathcal{P}(A)X = \sum_i \text{tr}[A^i X] |i\rangle$ be the map from the virtual operators X to the physical system of one site, let $\mathcal{P}(A^\dagger)|i\rangle = (A^i)^\dagger$ be the corresponding map for the tensor $A^\dagger$, and let $\sigma(X) = |G|^{-1} \sum_{g \in G} L_g X L_g^{-1}$ be the twirl onto the commutant of the left-regular representation. The formal notion of G -isometry allows $\mathcal{P}(A^\dagger)\mathcal{P}(A) = c\sigma$ with one constant $c > 0$ [con26]; then $P = c_2 \Pi_{\mathcal{S}_2}$ for the constant $c_2 > 0$ of the two-site tensor $A^i A^j$. The formalization therefore states

$$h = 1 - c_2^{-1}P, \quad (3)$$

together with $(c_2^{-1}P)^2 = c_2^{-1}P$, $\text{range } P = \mathcal{S}_2$ and $P^\dagger = P$. The last identity holds for every tensor, since the matrix entries of P satisfy $\overline{\text{tr}[A^k A^l (A^i A^j)^\dagger]} = \text{tr}[A^i A^j (A^k A^l)^\dagger]$. The source’s closing remark, that with A^{-1} in place of $A^\dagger$ the operator is a projector onto $\mathcal{S}_2$ that is not self-adjoint, is not formalized.

3 The proof of the commutation

Applying $\mathcal{P}(A^\dagger)\mathcal{P}(A) = c\sigma$ to an operator N that commutes with every L_g , for which $\sigma(N) = N$, gives

$$\sum_b \text{tr}[A^b N] (A^b)^\dagger = cN. \quad (4)$$

The tensor A is invariant, $L_g A^i L_g^{-1} = A^i$, and since L_g is unitary in the basis of group elements, $L_g^\dagger = L_{g^{-1}}$, each adjoint $(A^i)^\dagger$ also commutes with every L_g . Every product of the matrices A^i and $(A^i)^\dagger$ therefore commutes with every L_g ; this is the step in which the sum over g of figure “ham-comm-step1” drops out. Multiplying (4) on the left by any operator M and taking the trace gives the trace pairing

$$\sum_b \text{tr}[M(A^b)^\dagger] \text{tr}[A^b N] = c \text{tr}[MN]. \quad (5)$$

The matrix entry of $(P \otimes 1)(1 \otimes P)$ at the configurations $s_0 s_1 s_2$ and $w_0 w_1 w_2$ is computed with $M = (A^{w_0})^\dagger A^{s_0} A^{s_1}$ and $N = A^{s_2} (A^{w_2})^\dagger (A^{w_1})^\dagger$, which com-

³Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1177–1183 and 1534–1539.

commutes with every L_g . By cyclicity of the trace, and then by (5),

$$\begin{aligned} \sum_b \text{tr}[A^{s_0} A^{s_1} (A^{w_0} A^b)^\dagger] \text{tr}[A^b A^{s_2} (A^{w_1} A^{w_2})^\dagger] &= \sum_b \text{tr}[M(A^b)^\dagger] \text{tr}[A^b N] \\ &= c \text{tr}[MN] = c \text{tr}[A^{s_0} A^{s_1} A^{s_2} (A^{w_0} A^{w_1} A^{w_2})^\dagger]. \end{aligned} \quad (6)$$

For $(1 \otimes P)(P \otimes 1)$ the same computation applies with $M = A^{s_1} A^{s_2} (A^{w_2})^\dagger$ and $N = (A^{w_1})^\dagger (A^{w_0})^\dagger A^{s_0}$. The two matrices agree, so both products are c times the three-site operator of the same form. The two translated terms on any three consecutive sites of a ring commute, and terms with disjoint supports commute trivially.

4 Two-dimensional regional projectors

For a finite simple graph, assign a group label $\alpha_{v,e}$ to each half-edge. The actual canonical regular open-region range consists exactly of functions invariant under independent left multiplication at the region vertices and simultaneous right multiplication at the ends of each internal edge, and vanishing unless there is a vertex assignment k with $k_v^{-1} \alpha_{v,e} = k_w^{-1} \alpha_{w,e}$ on every internal edge $e = (v, w)$. This equality is proved from the original open-region contraction, including all independent dangling boundary labels; neither a rank formula nor connectedness is assumed.

Let A_v average the left action at vertex v , B_e average the shared right action on edge e , and D_R be the diagonal indicator of this flat locus on a region R . The extended actual regional range projector is

$$Q_R = \left(\prod_{v \in R} A_v \right) \left(\prod_{e \subseteq R} B_e \right) D_R.$$

Left and right multiplication commute. Averages on distinct vertices or edges act on distinct half-edge coordinates; coincident averages are identical. Flatness is invariant under every left vertex action and shared right edge action, and the D_R are diagonal. Thus $Q_R Q_S = Q_S Q_R$ for arbitrary overlapping regions.

For the original physical site maps, local G -isometry gives $A^\dagger A = cE$ with $c > 0$ and E the regular invariant projector. A supported orthogonal projector Q is transported to $c^{-1} A Q A^\dagger$, whose range is exactly the original physical regional range. Its intertwining relation is $PA = AQ$. No surjectivity onto the ambient physical space is imposed. The normalized local adjoints give nonzero physical site-image projectors J_v ; nonzero follows from the constant vector in the regular invariant subspace.

The physical regional projectors commute with every product of site-image projectors. For regions R, S , their commutator is supported on $R \cup S$ and absorbs the product of site-image projectors there. Its product with the full product of site maps is zero by the virtual commutation. Multiplying by the product of

normalized adjoints and canceling the nonzero complementary site-image factor therefore makes the commutator zero on the entire physical space, including unused local physical directions. The parent interactions are the complements of these range projectors and consequently commute.⁴

5 Scope and verdict

The two-dimensional commutation assertion is now proved for the original canonical physical parent interactions on every finite simple graph and arbitrary overlapping regions. In particular, it covers plaquette terms on ordinary square tori represented by this graph convention. The source’s projector-complement error is corrected as in (3).

A residual indexing restriction remains: the simple-graph description does not encode self-loops or parallel bonds in degenerate periodic presentations. The older matrix-product theorem still states $N \geq 3$ and does not by itself cover the two-site periodic double-bond convention. Square tori with both periods at least three have no such issue. No assertion about those smaller periodic presentations is inferred from the finite-simple-graph theorem. Eliminating this last restriction requires an incidence description retaining multiple half-edge occurrences, or a direct proof for the exceptional periodic contractions; it does not require another two-dimensional range or physical-transport hypothesis.

References

- [con26] The TNLearn contributors. Quantum-double PEPS: Normalization, representations and scope. Paper-gap note `rmp-peps-quantum-double-g-isometry`, TNLearn, 2026. <https://sirui-lu.com/TNLearn/paper-gaps/rmp-peps-quantum-double-g-isometry.pdf>.
- [SCPG10] Norbert Schuch, J. Ignacio Cirac, and David Pérez-García. PEPS as ground states: Degeneracy and topology. *Annals of Physics*, 325(10):2153–2192, 2010.

⁴The source-facing statement is `regularIsometric.canonicalRegionParentInteractions commute` in `TNLearn/PEPS/ParentHamiltonian/GIsometricRegionParentHamiltonian.lean`. Its hypotheses are local regular G -isometry; all support, nonzero-image, and regional intertwining identities are derived.