

Finite dual-path deformation of regular PEPS flux strings

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Verdict: scope-restriction (open).

1 Source and orientation

The source is Schuch, Cirac, and Pérez-García, arXiv:1001.3807, Definition 6.13 and Lemma 6.14, local source `Papers/1001.3807/paper_v3.tex`, lines 2174–2214. The endpoint measurement is Theorem 6.15, lines 2216–2267; lines 2230–2250 explain the four-site measurement and its independence from the outgoing virtual indices. The figures `figs5/def-flux-pair.pdf` and `figs5/detect-flux-end.pdf` fix the native orientation. This note belongs to GitHub issue #8703, under GitHub issue #8271.

Horizontal native arrows point right and vertical arrows point down. The source’s eastward dual segment puts U_g on its crossed downward bond; a southward dual segment puts $U_{g^{-1}}$ on its crossed rightward bond. The conjugacy labels of the initial and final plaquettes are consequently $[g]$ and $[g^{-1}]$ in the clockwise convention. The pre-existing graph detector follows the opposite, counterclockwise plaquette walk. It reads the inverse classes. This is an orientation change, not an equality between a conjugacy class and its inverse.

2 Paths and the derived gauge

Let $X = \mathbb{Z}_w \times \mathbb{Z}_h$ with $w, h > 0$. A plaquette is labelled by its lower-left primal vertex. A finite dual path is an actual sequence of east, west, north and south steps in the standard path construction of a directed graph. Distinct native bond labels remain distinct at periods one and two. Let $n(p) = (n_h(p), n_v(p))$ be the signed integer crossing counts. An east step from (x, y) contributes $(0, \delta_{(x+1, y)})$; a north step contributes $(\delta_{(x, y+1)}, 0)$; reversed steps contribute their negatives. Crossings add with multiplicity.

For $R \subseteq X$, the relation $p \sim_R q$ is generated by adjacent backtrack removal and the exchange of the east–north and north–east sides of a dual square enclosing a vertex of R . It is closed under symmetry, transitivity, reversal and

concatenation. It is defined without any premise about a PEPS state, a connection, a gauge, or flatness. Induction on these moves derives an integer potential $f : X \rightarrow \mathbb{Z}$ vanishing outside R with

$$n_h(q)(v) - n_h(p)(v) = f(v + (1, 0)) - f(v), \quad (1)$$

$$n_v(q)(v) - n_v(p)(v) = f(v) - f(v + (0, 1)). \quad (2)$$

For the positive square move, f is the point mass at the enclosed primal vertex. Reversing a move negates its potential and composing moves adds potentials. Retracing a whole path and removing a dual-square loop are consequences of these generators.

For any group G and $g \in G$, the inserted native group labels are $u_h^p(v) = g^{n_h(p)(v)}$ and $u_v^p(v) = g^{n_v(p)(v)}$. Taking $k(v) = g^{f(v)}$ in (2) derives

$$u_h^q(v) = k(v + (1, 0))u_h^p(v)k(v)^{-1}, \quad (3)$$

$$u_v^q(v) = k(v)u_v^p(v)k(v + (0, 1))^{-1}. \quad (4)$$

All products in this calculation are powers of the same element; G need not be abelian.

3 Actual physical equality and the exterior

For a finite-dimensional matrix representation U and site tensors A_v which are virtually G -invariant on R , the gauge in (4) is absorbed in the actual native bond-end sum. The physical coefficient is unchanged for every physical configuration. The site tensors may be inhomogeneous, their physical alphabets may vary, and site tensors outside R are unrestricted. No regularity, unitarity or injectivity is needed for this deformation argument beyond virtual invariance at the swept sites.

The exterior statement has an exact boundary. On every bond incident to R , use the path's prescribed matrix $U_{u_e^p}$ or $U_{u_e^q}$. On every bond whose two endpoints lie outside R , retain an arbitrary fixed matrix E_e . These exterior matrices need not be representation matrices and need not commute with U_g . Since $k = 1$ at both ends of such a bond, its matrix is unchanged. Crossing bonds are part of the vertex star and receive the derived gauge change. This proves equality of the actual closed contractions, rather than assuming a state equality or asserting equality for fixed open boundary columns.

The main declaration is `TNLean.PEPS.TorusDualHomotopy.torusBondNetwork_eq_with_exterior`. Its geometric input is `TNLean.PEPS.TorusDualHomotopy.exists_potential`; the contraction argument is `TNLean.PEPS.torusBondNetwork_groupGauge_of_invariant_on`. The state-valued specialization is `TNLean.PEPS.TorusDualHomotopy.torusDualFluxState_eq`.

4 Endpoint flux and measurement conventions

For native labels $u = (u_h, u_v)$, clockwise holonomy is

$$H_u(x, y) = u_h(x, y)^{-1} u_v(x + 1, y) u_h(x, y + 1) u_v(x, y)^{-1}. \quad (5)$$

The signed curl is the sum of the top and right crossing counts minus the bottom and left crossing counts. The single-step calculation telescopes along every path $p : a \rightarrow b$:

$$(\text{curl } n(p))(z) = \delta_a(z) - \delta_b(z), \quad (6)$$

$$H_{u^p}(z) = g^{\delta_a(z) - \delta_b(z)}. \quad (7)$$

Thus distinct endpoints carry g, g^{-1} and all other plaquettes have identity holonomy. If $a = b$, the holonomy is identity everywhere, even for a winding loop. Local absence of flux does not identify its global state with the state having no inserted string.

For $w, h \geq 3$, the native regular-matrix contraction is identified entrywise with the ordered-edge regular graph contraction, including at both seams. The fixed complete projective measurement on the four original physical spins at z therefore reads $[H_{u^p}(z)^{-1}]$ in the counterclockwise convention. Inversion gives the canonical bijection $\iota([s]) = [s^{-1}]$ of conjugacy classes. Relabelling the outcomes by $Q_C^{\text{cw}} = Q_{\iota(C)}^{\text{ccw}}$ gives the source's $[g]$ and $[g^{-1}]$ at the initial and final plaquettes. The extra orthogonal outcome is unchanged. All outcomes remain positive, orthogonal and complete; the projectors are independent of p and g .

The source-oriented measurement is `TNLean.PEPS.IsGIsometric.exists_torusDualFlux_clockwise_cutMeasurement`. It acts on the cut matrix formed from the actual inserted contraction. Regular G-isometry also implies that every such native string state and every physical cut matrix are nonzero, by the nonvanishing theorem for regular twisted graph states and the entrywise contraction identity. The input to endpoint detection is therefore proved nonzero. In the cyclic group C_3 , a generator and its inverse lie in different conjugacy classes. That example distinguishes the two conventions; S_3 alone cannot, since every permutation in S_3 is conjugate to its inverse.

5 Local parent consequences

Suppose $w, h \geq 3$ and the original homogeneous tensor is virtually G -invariant. Identity holonomy on one plaquette gives a gauge supported on its four sites which removes the four internal insertions. Every complementary physical slice of the actual inserted state then lies in the original plaquette PEPS range. Hence every positive parent interaction with that range as kernel annihilates the state there, including the canonical orthogonal projector onto the range's orthogonal complement. Substituting (7) derives these constraints at every nonendpoint plaquette, with no assumed vanishing curl or holonomy in the final path statement. A closed loop satisfies all such local parent constraints.

This is a range and local-annihilation statement; it does not assert equality of all reduced density matrices away from the endpoints.

6 Remaining scope and continuation

The established equality is for finite paths related by the specified local moves on a closed labelled torus. The lifted integer displacement reduces modulo the two periods to the endpoint difference and is invariant under those moves. Thus paths with the same periodic endpoints but different winding are not identified. The result does not classify all generated homotopy classes, prove that every equal-winding pair is related, or turn endpoint agreement alone into equality of states. The source lemma’s deformation language must be read with its topology and allowed local deformations specified.

The geometry and contraction identity hold for every positive period. The four-site detector and local parent consequences use a simple graph and require both periods at least three. A labelled-graph treatment of loops and parallel bonds is needed to extend those physical statements to smaller periods. Other surfaces and open boundary conditions require a corresponding path geometry and explicit boundary contraction theorem. No unrestricted version of Lemma 6.14 is marked as proved by this packet. The continuation under GitHub issue #8703 and GitHub issue #8271 is to extend these precisely stated boundaries when the wider source setting is required, rather than introduce a flatness assumption or a supplied gauge witness into the geometric deformation theorem.

7 Finite rectangular continuation

The finite bulk case of the deformation holds for embedded rectangles. Choose a translated lifted rectangle $[0, W] \times [0, H]$ with $W < w$ and $H < h$, retain the four unit-step labels, and require both paths to have the same lifted endpoints. Then the two projected paths are related by square moves and backtracks, and the two actual contractions with these paths are equal.¹ A bottom-row and column comb reduces the paths by adjacent-square exchanges and backtrack cancellations. The permitted swept primal sites form the whole rectangular interior $R = \{(x_0 + i + 1, y_0 + j + 1) : i < W, j < H\}$, with coordinates reduced modulo the two periods. The contraction statement requires virtual invariance at every site of this chosen R . Arbitrary fixed exterior matrices are allowed only on bonds whose two endpoints both lie outside R .

This specialization does not assert deformation avoiding an arbitrary region, equality of winding classes, an open-boundary theorem, or equality with a vacuum reduced density matrix. The unrestricted source gap remains open.

¹Lean statements: `TNLean.PEPS.TorusDualRectangle.homotopy` and `TNLean.PEPS.TorusDualRectangle.torusBondNetwork_eq_with_exterior`; see PR #8726.

8 Collared open-boundary continuation

Draft PR #8729 proves a further specialization with a one-site collar: $W + 1 < w$ and $H + 1 < h$. Its open coefficient is an independent finite sum over region incidences, with boundary indices fixed on crossing bonds and matrices on internal labelled bonds. A delta-exterior completion identifies this sum with a closed contraction exactly, after fixing the region physical indices and using a singleton physical alphabet outside. The collar makes a derived supported gauge trivial at crossing endpoints, which gives coefficient equality and contraction with an arbitrary joint boundary tensor. Strict compilation and eight axiom audits passed. The final theorem permits an empty virtual alphabet; the explicit full-region two-by-two regression has an empty boundary and a zero open coefficient. The unrestricted source gap, arbitrary-region avoidance, cross-winding equality and vacuum reduced-density conclusions remain open.