Deferred Result: The Translation-Invariant Bond-Dimension Lower Bound for the W State

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Verdict: open-gap (open).

1 The source statement

The review [CPGSV21] presents the W state

$$|W\rangle = |10\cdots 0\rangle + |010\cdots 0\rangle + \cdots + |0\cdots 01\rangle \tag{1}$$

on N sites as an open-boundary matrix product state of bond dimension $D = 2$, with the site-independent tensor

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \text{([CPGSV21, lines 2358–2360])}$$

and boundary covectors $(l| = (0|$ and $|r) = |1)$. It then observes that this is not a translation-invariant representation, and states the following bound [CPGSV21, line 2362]: any translation-invariant matrix product state whose periodic contraction equals $|W\rangle$ on N sites must have a bond dimension D that grows with the system size, satisfying a bound of the form

$$D^3 \log D = \Omega(N), \tag{2}$$

which in particular forces $D = \Omega\big(N^{1/(3+\delta)}\big)$ for every $\delta > 0$.

2 Why this is a separate, deferred theorem

The bound (2) is not proved in the review. The review obtains it by combining two external results:

1. the matrix product state representation theory of [PGVWC07], which controls when a state on N sites admits a translation-invariant representation at a given bond dimension; and

2. the quantum version of Wielandt's inequality, whose sharp D^2-scaling form (after [SPGWC10] and the subsequent sharpening attributed in the review to Michałek and Shitov) bounds the blocking length at which a primitive tensor's products span the full matrix algebra.

The conclusion is asymptotic in N: it constrains the growth rate of the optimal bond dimension, not a single finite-size identity. Establishing it in the formal development would require:

- a faithful statement of the translation-invariant representability criterion of [PGVWC07] for a prescribed finite-size state;
- the quantitative quantum Wielandt inequality with its explicit D^2 (and the refined $D^3 \log D$) scaling, beyond the qualitative primitivity and span-growth statements presently available in the `Wielandt` development;
- the asymptotic combination of the two into the growth bound (2), including the rank/dimension argument specific to the single-excitation symmetric state.

Each of these is a substantial independent development; together they are a multi-step effort well beyond the open-boundary identification recorded in the example file.

3 What is formalized

The example file establishes the open-boundary side of the source statement. It defines the site-independent W tensor, the boundary covectors, and the open-boundary contraction, and proves the central identification: for every N, the open-boundary amplitudes

$$(0|\, A^{\sigma_1} \cdots A^{\sigma_N}\, |1) \tag{3}$$

equal the indicator of the single-excitation configurations, so the open-boundary state is exactly the (unnormalized) W state.[1]

The translation-invariant lower bound (2) is *not* formalized and is *not* asserted as an established fact anywhere in the development.

4 Verdict

This is a deferred result, not a gap in a claimed formalization. The open-boundary $D = 2$ representation is fully proved. The translation-invariant bond-dimension lower bound is recorded here as an unformalized source statement requiring the representation theory of [PGVWC07] and the quantitative quantum Wielandt inequality; it should be tracked as a follow-up once that quantitative machinery is available.

[1] Formal declarations: `MPSTensor.wTensor`, `MPSTensor.openState`, `MPSTensor.wTensor_openState_eq_wIndicator`.

References

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.

[PGVWC07] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Matrix product state representations. *Quantum Information and Computation*, 7(5–6):401–430, 2007.

[SPGWC10] M. Sanz, D. Pérez-García, M. M. Wolf, and J. I. Cirac. A quantum version of Wielandt's inequality. *IEEE Transactions on Information Theory*, 56(9):4668–4673, 2010.