

SPT Fixed Points with a Nontrivial Character

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Verdict: scope-restriction (resolved).

1 The source statement

In Section III.A of [CPGSV21], a symmetry acts on an MPS by $S_g(A^i) = e^{i\theta(g)} X_g^\dagger A^i X_g$, where $\phi(g) = e^{i\theta(g)}$ is a circle-valued character. The source constructs a zero-correlation-length representative for a prescribed character and 2-cocycle class. On a ring of N sites, the character contributes the physical phase $\phi(g)^N$. The relevant passage is `Papers/2011.12127/TN-Review-main.tex`, lines 1147–1157.

2 The earlier restriction

The original exact-invariance predicate required each physical twist to have the same matrix product vectors as the untwisted tensor. Consequently, the fixed-point symmetry and class-realization results using that predicate were restricted to $\phi = 1$, although the local twist identity was already proved for arbitrary ϕ . Exact vector invariance omits the character phase when $\phi(g)^N \neq 1$.

3 Character and cocycle-class realization

Let ρ be a projective representation with factor system ω , put $W_g = \rho(g^{-1})$, and take the dimer fixed point

$$A^{(a,b)} = D^{-1/2} |a\rangle\langle b|, \quad U(g) = \phi(g)(W_g^T \otimes W_g^{-1}). \quad (1)$$

The symmetry relation is

$$\tilde{A}_g^i = \sum_j U(g)_{ij} A^j = \phi(g) W_g A^i W_g^{-1}. \quad (2)$$

Thus the twist is gauge equivalent to $\phi(g)A$, and

$$|V^{(N)}(\tilde{A}_g)\rangle = \phi(g)^N |V^{(N)}(A)\rangle. \quad (3)$$

These statements use on-site symmetry up to a character, rather than requiring equality of the two vectors. The projective representation ρ itself realizes (2). For $D \geq 1$, every other virtual projective representation realizing that relation with the same character has a factor system cohomologous to ω : cancel the nonzero scalar $\phi(g)$, then apply gauge uniqueness to the injective fixed-point tensor.

For finite G , a unitary twisted regular representation supplies the virtual representation from the cocycle alone. Its factor system is cohomologous to ω , and the physical action in (1) realizes the prescribed character and cocycle class. No unit-modulus assumption on a complex-valued character is needed here. Every g has finite positive order m , so $\phi(g)^m = 1$ and $|\phi(g)| = 1$ follow from the character law. For a general group, the algebraic identities hold for every complex-valued character; unitarity of the physical action requires the source's circle-valued character and unitary virtual matrices.

The general-character results are in `TNLean/MPS/Symmetry/Character` and `TNLean/MPS/Symmetry/SPTFixedPoint`. They include the vector phase, on-site symmetry up to the character, virtual class uniqueness and existence, and finite-group realization from the cocycle alone. The trivial-character restriction is therefore resolved.

4 Distinct source questions

This correction concerns the character phase. The use of the dimer fixed point in place of the printed tensor remains the local correction recorded in [con26a]. The action above treats linear on-site symmetries. Its mixed-symmetry extension includes time reversal and reflection, with a circle-valued character twisted by time reversal and a virtual factor system twisted by the sum of both parities [con26b]. In that extension, `MPSTensor.mixedSptAction_mpv` proves the same length- N phase $\phi(g)^N$, including complex conjugation and site-order reversal. The printed-tensor correction remains necessary in both constructions.

References

- [con26a] The TNLean contributors. SPT fixed points: The printed tensor is not normal. Paper-gap note `rm_p_spt_fixed_point_tensor`, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/rm_p_spt_fixed_point_tensor.pdf.
- [con26b] The TNLean contributors. SPT fixed points with time reversal and reflection. Paper-gap note `rm_p_spt_fixed_point_onsite_scope`, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/rm_p_spt_fixed_point_onsite_scope.pdf.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair

states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.