

SPT Fixed Points: The Printed Tensor Is Not Normal

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Verdict: false-source (resolved).

1 What the source prints

In the paragraph “Symmetry protected topological order” of [CPGSV21] a representative MPS with zero correlation length is constructed from a 2-cocycle ω and a 1-cocycle ϕ of a finite group G . The physical space is $\mathbb{C}^G \otimes \mathbb{C}^G$ with basis $|a, b\rangle$, the bond space is $\mathbb{C}^G$, and the source prints

$$A_{xy}^{ab} = e^{i\alpha(ax)\omega(a,x)+i\beta(b)\phi(b)} \delta_{y,ax}, \quad (X_g)_{xy} = e^{i\alpha(x)\omega(x,g)} \delta_{y,xg}. \quad (1)$$

Here α and β are the source’s sign morphisms $G \rightarrow \{\pm 1\}$ recording which elements act by time reversal or reflection; for an on-site symmetry both are trivial. It states that the relation $S_g(A^{ab}) = e^{i\phi(g)} X_g^\dagger A^{ab} X_g$ is equivalent to the 2-cocycle equation, that the transfer matrix is a rank-one projector, and that for an on-site $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry the construction yields the cluster state after blocking two sites.¹

2 The printed tensor is not normal

Take the on-site case, α and β trivial, and write ω and ϕ multiplicatively. Then $A^{ab} = \phi(b)L_a$ with $(L_a)_{xy} = \omega(a,x)\delta_{y,ax}$, a twisted left-regular matrix, and X_g is a twisted right-regular matrix. A direct computation gives

$$(L_a X_g)_{xz} = \omega(a,x)\omega(ax,g)\delta_{z,axg}, \quad (X_g L_a)_{xz} = \omega(x,g)\omega(a,xg)\delta_{z,axg}, \quad (2)$$

so $L_a X_g = X_g L_a$ for all a, g exactly when ω satisfies the 2-cocycle equation $\omega(a,x)\omega(ax,g) = \omega(a,xg)\omega(x,g)$. This is the equivalence the source states, but it shows that every X_g commutes with every letter, so $X_g^\dagger A^{ab} X_g = A^{ab}$. For $g \neq 1$ the matrix X_g has no nonzero diagonal entry and is not a scalar, so it commutes with every product of letters, and no space of words is the full matrix

¹Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 1147–1157.

algebra: the tensor is not normal once $|G| > 1$. The transfer map is not of rank one either: $\sum_{a,b} A^{ab} M A^{ab\dagger} = |G| \sum_a L_a M L_a^\dagger$ sends each X_g to $|G|^2 X_g$, and the $|G|$ matrices X_g have disjoint supports, so the largest eigenvalue $|G|^2$ has multiplicity at least $|G|$. For instance, for $G = \mathbb{Z}_2$ with ω and ϕ trivial the four letters are $1, 1, \sigma_x, \sigma_x$, which generate a commutative algebra. The printed claims thus fail on nondegenerate data, and the note is filed as a false source claim.

3 The formalized fixed point

The claims hold for the dimer fixed point that the paragraph describes. Let ρ be a projective representation of G on $\mathbb{C}^D$, $D \geq 1$, with factor system ω , and let ϕ be a character. Put $W_g = \rho(g^{-1})$ and

$$A^{(a,b)} = D^{-1/2} |a\rangle \langle b|, \quad U(g) = \phi(g) W_g^T \otimes W_g^{-1}. \quad (3)$$

The factor system cancels between the two tensor factors, so U is a linear representation, and $\sum_j U(g)_{ij} A^j = \phi(g) W_g A^i W_g^{-1}$. The transfer map is $M \mapsto D^{-1} \text{tr}(M) 1$, a rank-one projector, and the letters span the full matrix algebra. For $\phi = 1$ the tensor has MPV-family invariance under U , and every virtual representation it induces has a factor system cohomologous to ω . Choosing for ρ the twisted regular representation on $\mathbb{C}^G$ gives the dimensions of the printed construction. For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the two-dimensional Pauli representation, the tensor is the length-two blocked cluster tensor up to a unitary on the physical space.

4 Formal boundary

The on-site construction described above takes a projective representation with factor system ω as input; for finite groups such a representation, with factor system in the class of ω , is supplied by the twisted regular representation, as recorded in [con26a]. The formal construction is `MPSTensor.sptFixedPointTensor` with the action `MPSTensor.sptFixedPointAction`; the cluster comparison is `MPSTensor.clusterBlocked_eq_sum_sptFixedPointTensor`. The mixed-symmetry extension now includes time reversal and reflection, with phase action $\beta = t$ and virtual action $\alpha = t + r$, while retaining the same corrected dimer tensor [con26b].

References

- [con26a] The TNLearn contributors. SPT fixed points from a supplied projective representation. Paper-gap note `rmp_spt_fixed_point_supplied_representation`, TNLearn, 2026. https://sirui-lu.com/TNLearn/paper-gaps/rmp_spt_fixed_point_supplied_representation.pdf.

- [con26b] The TNLean contributors. SPT fixed points with time reversal and reflection. Paper-gap note `rm_p_spt_fixed_point_onsite_scope`, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/rm_p_spt_fixed_point_onsite_scope.pdf.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.