

SPT Fixed Points with Time Reversal and Reflection

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Verdict: scope-restriction (resolved).

1 What the source states

The symmetry group in Section III.A of [CPGSV21] may combine on-site, time-reversal and reflection symmetries. Write $t, r : G \rightarrow \mathbb{Z}_2$ for the latter two parities and let C_p be complex conjugation at odd parity and the identity at even parity. The local symmetry is $S_g(A)^i = \phi(g)X_g^\dagger A^i X_g$. Its phase is a one-cocycle for $\beta = t$, whereas its virtual factor system is a two-cocycle for $\alpha = t + r$. The source asserts that every such pair has a zero-correlation-length representative.¹

2 The resolved scope

The earlier formalization handled linear on-site symmetries only. `MPSTensor.mixedTensorAction` now conjugates tensor letters for time reversal and transposes them for reflection. The ordered-product theorem `TNLean.Algebra.symmetryLetter_list_prod` and `MPSTensor.mpv_mixedTensorAction` show that reflection reverses the actual finite-chain site order. Both effects are included simultaneously.

The coefficient group is the existing unitary scalar group, literally $U(1)$, with its conjugation action. The classes `TNLean.Algebra.parityUnitaryH1Class` and `TNLean.Algebra.parityUnitaryH2Class` use Mathlib's group cohomology; they are not a second hand-built quotient or a relabeling of all complex units. The equations are

$$\phi(gh) = \phi(g)C_{t(g)}(\phi(h)), \quad (1)$$

$$\omega(gh, k)\omega(g, h) = C_{t(g)+r(g)}(\omega(h, k))\omega(g, hk). \quad (2)$$

The two parities are deliberately distinct: reflection conjugates the virtual gauge through transposition, but does not conjugate a scalar phase.

¹Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 1120–1130 and 1147–1157.

For finite G , the unitary regular matrices $L_g(x, y) = \omega(g, y)\delta_{x, gy}$ realize the chosen factor system exactly. `TNLean.Algebra.conjugateRegularUnitaryRepresentation` does not assume that the cocycle representative is normalized. The cocycle equation itself gives $L_g C_{t(g)+r(g)}(L_h) = \omega(g, h)L_{gh}$.

Using these virtual matrices, the dimer letters $A^{(a,b)} = D^{-1/2}|a\rangle\langle b|$, $D = |G|$, carry the physical action

$$U(g) = \phi(g)(\overline{X_g} \otimes X_g)P^{r(g)}, \quad (3)$$

$$\sum_j U(g)_{ij} T_{t(g), r(g)}(A^j) = \phi(g) X_g^\dagger A^i X_g, \quad (4)$$

where P swaps the two dimer factors and $T_{t,r}$ is entrywise conjugation followed by transpose when the corresponding parity is odd. `MPSTensor.exists_mixed_spt_fixedPoint` proves injectivity, the rank-one idempotent transfer map, physical unitarity, the semilinear group law and the prescribed tensor symmetry. The physical matrices satisfy $U(gh) = U(g)C_{t(g)}(U(h))$; conjugation is applied separately when the symmetry is antiunitary. The finite-chain vector has phase $\phi(g)^N$, as proved by `MPSTensor.mixedSptAction_mpv`.

`MPSTensor.parityUnitaryH2Class_eq_of_mixedSptSymmetry` proves that any other unitary virtual realization of the same physical symmetry and character has the same $U(1)$ cohomology class. Injectivity gives pointwise scalar gauge equivalence, unitarity forces unit-modulus scalars, and their ratio is the twisted coboundary.

3 Conventions and remaining boundaries

At $t = r = 0$ the local action and cocycle equations specialize to the on-site ones. The existing arbitrary-character on-site theorems remain available. The construction retains the corrected dimer tensor: it does not reinstate the erroneous printed tensor discussed in [con26]. Its bond space is nonzero because $D = |G|$ for a finite group. The result is the fixed-point realization and its class independence; it does not by itself prove a new mixed-symmetry gapped interpolation theorem.

References

- [con26] The TNLean contributors. SPT fixed points: The printed tensor is not normal. Paper-gap note `rmp_spt_fixed_point_tensor`, TNLean, 2026. https://sirui-lu.com/TNLean/paper-gaps/rmp_spt_fixed_point_tensor.pdf.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.