

Quantum-Double PEPS: Normalization, Representations and Scope

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Verdict: scope-restriction (open).

1 What the sources print

Definition “def:2d-Ug-inj” of [SCPG10] calls a PEPS tensor A G -injective for a semi-regular representation $g \mapsto U_g$, one containing every irreducible representation of the finite group G , if $\mathcal{P}(A)U_g = \mathcal{P}(A)$ and $\mathcal{P}(A)$ has a left inverse L with $L\mathcal{P}(A) = \Pi_U$, the projector onto the invariant subspace.¹ Definition “def:iso:isopeps” calls it G -isometric if moreover U_g is the left-regular representation $L_g|h\rangle = |gh\rangle$ and $\mathcal{P}(A)$, restricted to its domain and range, is unitary.² The quantum-double tensor

$$K = \sum_{p,q,r,s \in G} |pq^{-1}, qr^{-1}, rs^{-1}, sp^{-1}\rangle \otimes |p, q\rangle \langle r, s| \quad (1)$$

is then called G -isometric.³ Appendix A of [CPGSV21], under “The Toric Code and quantum double models”, gives the same tensor with plaquette colors on the virtual legs and the differences of adjacent colors as physical spins, states that it is G -isometric for the regular representation, and states that the virtual labels of the primal tensor fuse to the identity, so that the primal tensor has a symmetry under every irreducible representation.⁴

2 Normalization

Neither tensor is normalized. Each physical configuration of differences arises from exactly $|G|$ colorings, those related by one global shift $h_i \mapsto h_i g$. For vectors x, y invariant under the shift,

$$\langle Kx, Ky \rangle = |G| \langle x, y \rangle. \quad (2)$$

¹Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1010–1013 and 1278–1296.

²Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1692–1700.

³Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 2904–2911.

⁴Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2450–2465.

So K restricted to the invariant subspace is $\sqrt{|G|}$ times a unitary onto its range, and $K/\sqrt{|G|}$ satisfies the printed definition. The formal notion of G -isometry asks for $\langle Kx, Ky \rangle = c \langle x, y \rangle$ with one constant $c > 0$, which the rescaling by $c^{-1/2}$ turns into the printed notion; the quantum-double tensor satisfies it with $c = |G|$. For a matrix product tensor with the left-regular representation on its bond the formal notion is the first form of Definition “def:iso:isopeps”, $\mathcal{P}(A^\dagger)\mathcal{P}(A) = c\Pi$; under concatenation $C^{ij} = A^i B^j$ the constants multiply, $c_C = c_A c_B$, and for $c_A = c_B = 1$ this is the source’s Lemma “lemma:iso:iso-stable-under-concat”.⁵

2.1 The primal toric-code tensor

For the primal tensor in Appendix A of [CPGSV21], specialize to $G = \mathbb{Z}_2$ and write the group operation additively. Its map P from the four virtual legs to the four physical edge spins is

$$(Px)(g_1, g_2, g_3, g_4) = x(g_1 + g_2, g_2 + g_3, g_4 + g_3, g_1 + g_4). \quad (3)$$

Let $Z|a\rangle = (-1)^a|a\rangle$ and $S = Z^{\otimes 4}$ on the virtual legs. The invariant subspace of the representation of $\mathbb{Z}_2$ generated by S consists of vectors supported on even virtual configurations $t + r + b + l = 0$. Each such configuration has exactly two physical preimages under the label map in (3), whereas odd configurations have none. Consequently,

$$PS = P, \quad (4)$$

$$P^\dagger P = 2\Pi_{\text{even}}, \quad (5)$$

$$\Pi_{\text{even}} = \frac{I + S}{2}. \quad (6)$$

Thus P is injective on the invariant subspace and satisfies $\langle Px, Py \rangle = 2\langle x, y \rangle$ there; $P/\sqrt{2}$ is an isometry onto its range. This is a local $\mathbb{Z}_2$ consequence of the primal tensor (3), with its character representation on the virtual legs. The general-group primal symmetry is discussed below.⁶

3 The four primal toric-code torus vectors

The one-bond character representation is now explicit: $U_g = \text{diag}_{a \in \mathbb{Z}_2} (-1)^{ga}$, so $U_0 = I$ and $U_1 = Z$. Its oriented four-leg action is exactly the parity action above. The matrices I, Z are linearly independent, hence the representation is

⁵Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1704–1716.

⁶Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2451–2465, equation `eq:app:tcode-rep-primal`. Formal declarations in `TNLean/PEPS/Examples/ToricCodePrimal.lean`: `TNLean.PEPS.toricCodePrimalVirtualRep_generator`, `TNLean.PEPS.toricCodePrimalMatrix_conjTranspose_mul_self`, and `TNLean.PEPS.isGIsometric_toricCodePrimalTensor`.

semi-regular. Applying the general closure-independence theorem to the literal primal tensor gives four independent contracted vectors $\Psi_{g,h}$ and

$$\dim_{\mathbb{C}} \text{span}\{\Psi_{g,h} : g, h \in \mathbb{Z}_2\} = 4. \quad (7)$$

This conclusion concerns actual seam insertions in the tensor network, not four abstract labels.⁷

For a physical configuration, let (t_v, r_v, b_v, l_v) be its uniquely specified virtual labels. The amplitude vanishes unless $l_{v+(1,0)} = r_v$ and $b_{v+(0,1)} = t_v$ at every site. When these conditions hold, it equals the product of $(-1)^{hr_v}$ over the horizontal seam and $(-1)^{gt_v}$ over the vertical seam. Thus the untwisted amplitude is exactly one, and all four vectors have the same configuration support, with different relative signs. No tensor or state normalization is suppressed.⁸

The oriented-bond contraction is defined for all positive rectangular periods. For periods at least three it has the same four distinct incident bonds as the simple-graph PEPS convention. The smaller periods are algebraic extensions. For periods at least three the general uniform semi-regular parent theorem now supplies the full-kernel classification; the smaller-period parent model remains a separate assertion.

4 Representations

The formal notions of G -injectivity and G -isometry take the representation as a parameter and do not require it to be semi-regular or left-regular; each instance names the representation it uses. The quantum-double instance shifts every color by right multiplication, $h \mapsto hg$, which leaves the differences hh'^{-1} unchanged also for non-abelian G . On each leg this is the right-regular representation $R_g|h\rangle = |hg^{-1}\rangle$, and the unitary $|h\rangle \mapsto |h^{-1}\rangle$ intertwines it with the left-regular one. Both are semi-regular. The one-dimensional notion and the stability under concatenation fix the left-regular representation, since their proof uses $\Delta = |G|^{-1}\mathbb{K}$. The correspondence between virtual unitaries commuting with the symmetry and physical unitaries, Lemma “lemma:iso:sym-virt-can-be-done-on-phys”, uses only the isometry of $\mathcal{P}(A)$ on the invariant subspace, and is stated for the representation parameter.⁹

5 Concatenation in one and two dimensions

Lemma “lemma:iso:iso-stable-under-concat” states that G -isometry is stable under concatenation of PEPS; its proof is written for the one-dimensional left inverse of Lemma 4.7.¹⁰ The formalization proves the one-dimensional case for

⁷Results: `TNLean.PEPS.torusLegRep_toricCodeBondRep`, `TNLean.PEPS.isSemiRegular_toricCodeBondRep`, `TNLean.PEPS.linearIndependent_toricCodeTorusState`, `TNLean.PEPS.firank_span_toricCodeTorusState`.

⁸Result: `TNLean.PEPS.toricCodeTorusState_apply`.

⁹Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1729–1761.

¹⁰Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1688–1689 and 1704–1716.

matrix product tensors with the left-regular representation on the bond, and the two-dimensional case for the contraction of one link of two PEPS tensors with the left-regular representation on that link. In two dimensions the virtual system of the first tensor is $W_A \otimes \mathbb{C}[G]^*$ and that of the second is $\mathbb{C}[G] \otimes W_B$; in the basis of group elements on the link and coordinate bases of the remaining legs, the representations are $\sigma_A \otimes L$ and $L \otimes \sigma_B$, the contragredient of the permutation representation L having the same matrices in the dual basis. For a unitary representation, G -isometry with constant c is

$$\mathcal{P}(A)^\dagger \mathcal{P}(A) = c \Pi, \quad (8)$$

where Π is the projector onto the invariant subspace. The Gram matrix of the contracted tensor is the contraction of the two Gram matrices, and on the link the group elements g, h of the two projectors meet in $\sum_{k,k'} (L_g)_{kk'} (L_h)_{kk'} = |G| \delta_{g,h}$, which is the step $\Delta = |G|^{-1} \mathcal{K}$ of the source. Hence $\mathcal{P}(C)^\dagger \mathcal{P}(C) = c_A c_B \Pi$: the left inverse of the contracted tensor is $(c_A c_B)^{-1} \mathcal{P}(C)^\dagger$. Entrywise $\mathcal{P}(C)^\dagger_{(a,b),(k,l)} = \sum_g \mathcal{P}(A)^\dagger_{(a,g),k} \mathcal{P}(B)^\dagger_{(g,b),l}$, so this adjoint is the contraction of the adjoints of the two factors. The remaining legs are required to carry unitary representations; in the source every leg of a G -isometric tensor carries the left-regular representation or its contragredient, so this follows from the source's hypotheses. This item is resolved. The untwisted contraction of an arbitrary nonempty connected finite region is also formalized, including regions containing closed virtual cycles. If V_R and I_R count its vertices and internal edges, and $A_v^\dagger A_v = c_v \Pi_v$, its genuine open tensor satisfies

$$T_R^\dagger T_R = \left(\prod_{v \in R} c_v \right) \frac{|G|^{I_R+1}}{|G|^{V_R}} \Pi_{\partial R}. \quad (9)$$

The bonds are unnormalized, and no group operators are inserted on them. Expanding the local Gram identities yields a group label at each vertex. Equality along every internal edge and connectedness make these labels one common translation. Each internal label remains free, contributing one factor $|G|$; the remaining boundary average is $|G| \Pi_{\partial R}$. The boundary may be empty.¹¹

The actual block also satisfies the defining invariant-subspace isometry relation, with the positive factor of (9).¹² Its accessible physical coordinates are obtained explicitly. Applying the local map $c_v^{-1} A_v^\dagger$ at every vertex turns the original open contraction into the open contraction B_R of the averaging projectors Π_v .¹³ Applying the original A_v to these projector coordinates recovers the original contraction, since $A_v \Pi_v = A_v$.¹⁴ These are identities of the original finite sums, for arbitrary finite regions. They do not assert that the two rectangular maps are inverse on unused physical subspaces.

For a connected region, choose a spanning tree and a root. There is a reversible change from its physical half-edge labels to boundary labels y , one

¹¹Result: `TNLean.PEPS.regularOpenRegionMatrix_gram_of_connected`.

¹²Result: `TNLean.PEPS.isGIsometric_regularOpenRegionMatrix_of_connected`.

¹³Result: `TNLean.PEPS.regionPhysicalProductMatrix_mul_regularOpenRegionMatrix`.

¹⁴Result: `TNLean.PEPS.regionPhysicalMap_regularProjectorOpenRegionMatrix`.

reference label a_e per internal edge, normalized vertex labels r_v with $r_o = 1$, and one residual label z_e per edge outside the tree. In these coordinates the actual canonical block is

$$B_R(y, a, r, z; \theta) = |G|^{1-|R|} \Pi_{\partial R}(y, \theta) \mathbf{1}_{\{z_e=1 \text{ for all } e \notin T\}}. \quad (10)$$

Thus the reference and normalized vertex labels give uniform factors, while the residual cycle labels give fixed identity factors. This factorization is derived from the internal-bond summation, without assuming a block decomposition. It gives an explicit form of the accessible-coordinate and blocking arguments of [SCPG10].¹⁵

The same coefficient formula proves G -injectivity of a connected regular block from local G -injectivity alone. Choose local inverses $L_v A_v = \Pi_v$ and apply their product to the actual block. In the canonical coordinates, the row with $a_e = r_v = z_e = 1$ is the nonzero multiple $|G|^{1-|R|} \Pi_{\partial R}$. It therefore recovers every invariant boundary vector. This proves the regular-representation connected-region consequence of Lemma 5.2 and the following observation of [SCPG10], including internal cycles and an empty boundary. The general semi-regular connected-region conclusion is not asserted by this argument.¹⁶

For arbitrary inserted bond operators u_e , the same physical maps relate the actual open tensor to the canonical contraction $B_R^u(\alpha, \theta)$ in both directions. A normalized tree gauge $k_o = 1$, $k_h k_t^{-1} = u_e$ on tree edges, makes the corresponding residual operators $\omega_e = k_h^{-1} u_e k_t$ equal to the identity on the tree. The residual operators on the other internal edges remain unrestricted. The transported boundary label is $k_v^{-1} \theta_e$ at a tail endpoint and $k_v^{-1} u_e \theta_e$ at a head endpoint. These are actual coordinate identities; they do not assert that arbitrary internal twists preserve the untwisted boundary projector or its rank.¹⁷

In spanning-tree coordinates, the twisted canonical coefficient is

$$B_R^u(y, a, r, z; \theta) = |G|^{-|R|} \sum_{x \in G} \mathbf{1}_{\{y=x\theta'\}} \mathbf{1}_{\{z_e=x\omega_e x^{-1} \text{ for all } e \notin T\}}. \quad (11)$$

The internal reference and normalized vertex coordinates remain free. This identity includes arbitrary crossing operators, and follows from a bijection between compatible internal labelings and the surviving common translations x .¹⁸

More generally, an arbitrary vertex gauge changes the actual open tensor to the tensor with internal operators $k_t^{-1} u_e k_s$ and transported crossing labels. If all these internal residuals are identities, its ordered matrix has the form $T_R^u = T_R F$, where F is the boundary basis permutation. Local regular isometry and connectedness then give

$$(T_R^u)^\dagger T_R^u = c_R F^\dagger \Pi_{\partial R} F, \quad c_R > 0. \quad (12)$$

¹⁵Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1765–1920. Result: `TNLean.PEPS.regularProjectorOpenRegionMatrix_coordinates`.

¹⁶Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1318–1358. Result: `TNLean.PEPS.isGInjective_regularOpenRegionMatrix_of_connected`.

¹⁷Results: `TNLean.PEPS.regionPhysicalMap_eq_regularProjectorTwistedRegionMatrix`, `TNLean.PEPS.regularRegionGaugeResidual_eq_one_of_tree`.

¹⁸Result: `TNLean.PEPS.regularProjectorTwistedRegionMatrix_coordinates`.

The transported projector has rank and trace $|G|^{|\partial R|-1}$ for a nonempty boundary. The residual-flatness condition here is an explicit hypothesis; it is not a substitute for the source's geometric disk condition. On the closed lattice, the same endpoint gauge change leaves the original physical vector exactly unchanged, without any isometry assumption, by local regular invariance and reindexing the summed bond labels.¹⁹

For a native torus region with circumferences $w, H \geq 3$, a supplied integer lift $L : R \rightarrow \mathbb{Z}^2$ is required to project to each original vertex and to preserve every internal rightward and upward unit step. These are arithmetic equations independent of the group. For commuting closure labels, the explicit gauge

$$k_v = h^{\lfloor L_x(v)/w \rfloor} g^{-\lfloor L_y(v)/H \rfloor} \quad (13)$$

has the native closure operator as its gradient on every internal edge, including seam edges. Thus identity residuals, the positive transported Gram formula, and the boundary rank follow without an additional group-valued flatness hypothesis.²⁰ Normalizing this gauge at the root identifies it with the spanning-tree gauge. The actual canonical tensor then has the same normalized internal ancillary vector in every commuting sector; only its transported boundary configuration depends on the sector.

The supplied lift does not define a disk: some annular regions also admit lifts. Conversely, a filled plaquette is a disk whose induced vertex graph has a cycle, so acyclicity of that graph would impose the wrong geometric restriction. A native region is realized as the finite union of closed unit cells centered at its sites on the continuous torus. This union is compact and locally path connected: the finite coproduct of its compact convex squares maps onto it by a closed quotient map. Every internal native unit-edge path lies in the two endpoint cells, including across a periodic seam. If the realization is simply connected, the two additive-circle covering maps give a continuous lift to the real plane. Its values at native sites are integers, and uniqueness of path lifts gives exactly the required integer unit-step equations. Thus no arithmetic lift is an additional hypothesis in this simply connected case.²¹

The continuous lift is a closed embedding. On each native cell, it is translation of the closed unit square by the lifted integer site, and its image is exactly the finite union of these planar squares. Thus the original torus cell domain is homeomorphic to this actual planar cell union. This identifies the domain for an exterior path argument; it does not supply such a path.²²

For a graph-connected native region with simply connected cell realization, local regular isometry and commuting closure labels therefore imply that

¹⁹Results: `TNLean.PEPS.exists_positive_gram_regularTwistedOpenRegionMatrix_of_regularRegionGaugeResidual_eq_one`, `TNLean.PEPS.rank_regularTwistedOpenRegionMatrix_of_regularRegionGaugeResidual_eq_one`, `TNLean.PEPS.sameState_regularVertexGauge`.

²⁰Results: `TNLean.PEPS.IsGIsometric.exists_positive_gram_torusClosureRegionMatrix_of_integerLift`, `TNLean.PEPS.IsGIsometric.rank_torusClosureRegionMatrix_of_integerLift`.

²¹Result: `TNLean.PEPS.exists_isTorusRegionIntegerLift_of_isSimplyConnected`.

²²Results: `TNLean.PEPS.continuousLift_torusRegionRealization_isClosedEmbedding`, `TNLean.PEPS.range_continuousLift_torusRegionRealization`.

its actual open-region matrix has rank $|G|^{|\partial R|-1}$ whenever the boundary is nonempty.²³ Graph connectivity remains explicit: cells touching only at diagonal points can be connected as a geometric union without connecting the native induced graph. This auxiliary rank statement concerns one closure state; it is not the source’s full disk entropy theorem. The source’s complement-only controlled permutations must then remove the sector-dependent boundary transports simultaneously. Separate single-sector Gram identities do not establish a common reduced density operator for coherent superpositions.

The untwisted region Gram formula is the identity-operator case of these calculations. For arbitrary twisted open regions, the cycle and crossing compatibility constraints remain essential. The actual torus overlap calculation below retains the noncontractible closure operators, and does not justify discarding those constraints on an arbitrary open region. The residual small-period torus restriction, together with the remaining source results, are tracked in issue #8271.

6 Actual torus closure overlaps

The regular G -isometric torus closure vectors are contracted with the horizontal h -seam and the downward-oriented vertical g -seam of the closure definition in [SCPG10].²⁴ For N sites, allowing site-dependent normalization $c_v > 0$, their overlap is

$$\langle \psi_{g,h}, \psi_{g',h'} \rangle = \left(\prod_v c_v \right) |G|^N \# \{x \in G : hx = xh', gx = xg'\}. \quad (14)$$

This identity is derived from the actual tensor contraction. The local Gram averages contribute $|G|^{-N}$; the two families of free bond labels contribute $|G|^{2N}$. Every vertex translation is constant off the seams, and the seam equations are the two intertwining conditions in (14). The proof includes tori with a circumference of one.²⁵

For a self-overlap the count is $|Z_G(g) \cap Z_G(h)| > 0$, since it contains the identity. Different pair-conjugacy classes give orthogonal nonzero vectors. Consequently the family indexed by all pair-conjugacy classes is linearly independent, as is its commuting subfamily.²⁶ Noncommuting closures are not asserted to be ground states.

The isometry assumption is now removed for regular virtual representations. Let $T = \mathcal{P}(A)$ and let L be the local inverse furnished by G -injectivity, so that $LT = \Pi$. Applying L at every physical site sends the actual closure vector of A to that of the canonical site map Π . The latter is G -isometric, with factor

²³Result: `TNLean.PEPS.IsGIsometric.rank_torusClosureRegionMatrix_of_isSimplyConnected`.

²⁴Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1515–1524.

²⁵Result: `TNLean.PEPS.IsGIsometric.exists_torusGClosure_overlap`.

²⁶Results: `TNLean.PEPS.IsGIsometric.linearIndependent_torusGClosureClass`, `TNLean.PEPS.IsGIsometric.linearIndependent_torusGClosureClass_commuting`.

one, so its class-indexed closures are linearly independent. Applying the same physical product map to a linear relation proves independence of the original closures. In particular the commuting family satisfies the regular-representation independence assertion in [SCPG10, Theorem 5.9], with no isometry or physical surjectivity hypothesis.²⁷ The auxiliary extension to all pair classes still makes no ground-state assertion.

Conversely, applying T at every physical site to a projector closure recovers the original closure, since $T\Pi = T$. These two rectangular physical maps recover the corresponding actual vectors; they need not be inverses on the full physical spaces. A physical product map changes the coefficient matrix across any partition by $C \mapsto F_R C F_{R^c}^T$, so its rank cannot increase. Recovery in both directions therefore preserves the Schmidt rank.²⁸ This is the local deformation argument required in [SCPG10, Corollary 6.10]; it does not yet supply the source's boundary-rank formula on arbitrary contractible regions.

Independence is now also established for general semi-regular virtual representations. After the same local inverse operation, expanding the actual projector contraction gives one group label q_v per vertex. On every bond away from the seams, the weighted trace pairing with Δ gives $\delta_{q_v, q_{v'}}$. Thus all vertex labels coincide. On each seam retain one operator, and contract the remaining seam operators with the sum of the trace-dual functionals, which takes value one on every U_g . The resulting linear map E satisfies

$$E\psi_{g,h}^\Pi = |G|^{-N} \sum_{x \in G} U_{xhx^{-1}} \otimes U_{xgx^{-1}}. \quad (15)$$

These operator sums are independent when indexed by pair-conjugacy classes: the tensor product of the two trace-dual functionals vanishes on different classes and gives $|Z_G(g) \cap Z_G(h)| > 0$ on its own class. Since $|G|^{-N} \neq 0$, the actual closure vectors are independent.²⁹ This proves the independence component of [SCPG10, Theorem 5.9], with its semi-regular hypothesis, without a local isometry or a contracted-region Gram hypothesis. The construction includes noncommuting pairs as an auxiliary extension and tori of circumference one. Ground-space membership and spanning remain separate steps.

7 Scope of the physical entropy calculation

The normalized actual finite-graph PEPS has entropy $S(\rho_R) = (\partial R - 1) \log |G|$, rank $|G|^{\partial R - 1}$, and a flat reduced spectrum when every local tensor is G -isometric in its regular group coordinates, the induced graphs on R and its complement are nonempty and connected, the boundary is nonempty, and the contraction has no inserted bond operators. The two region Gram identities are derived

²⁷Results: `TNLean.PEPS.IsGInjective.linearIndependent_torusGClosureClass`, `TNLean.PEPS.IsGInjective.linearIndependent_torusGClosureClass_commuting`.

²⁸Result: `TNLean.PEPS.IsGInjective.rank_physicalCutMatrix_torusGClosure_sum`.

²⁹Result: `TNLean.PEPS.IsGInjective.linearIndependent_torusGClosureClass_of_isSemiRegular`.

from the local isometries; they are not additional hypotheses. Their positive factors give an explicit positive normalization of the actual coefficient vector.³⁰

Theorem 6.9 of [SCPG10] concerns the torus states with (g, h) -closures, including their superpositions. The finite untwisted statement above is therefore a restriction, not the formalization of that source theorem. The local controlled permutation $(a, b, c) \mapsto (cb^{-1}a, b, c)$ and its inverse have been established, including the cancellation of a twist t when $c = tb$.³¹ Accessible coordinates, exact twisted coefficients and mixed Gram matrices, and the factorization on supplied integer-lifted regions are established above. Simple connectedness of the cell realization now yields the geometric lift and eliminates the region's internal cycle operators. The remaining boundary transport must be removed by one operation on the complement, independent of the closure sector.

The complement's spanning-tree coordinates provide a concrete route. Write its cycle tuple as $z = x\omega x^{-1}$ on the support of the actual coefficient. Let m_f be the combined transport on crossing bond f , and fix a reference bond f_0 . The actual two boundary equivalences compose as $F_S \circ F_R^{-1}$, and their multiplier is $m_f = k_S(w_f)^{-1} D_f k_R(v_f)$, where $v_f \in R$, $w_f \in S$, and D_f transports from v_f to w_f . The actual closed route from the complementary root through f_0 backwards, the region tree, f forwards, and the complementary tree back to the root has holonomy $m_f m_{f_0}^{-1}$. This identity is established for arbitrary bond operators; the route passes through R .³²

For each actual induced complementary walk, assigning free generators to the non-tree edges constructs a word whose evaluation at ω is its reduced holonomy. At the normalized root it is its original holonomy. Simultaneous conjugation of the tuple conjugates the word's evaluation. The required geometric step is to replace the route through R by a complementary loop with the same torus seam crossing numbers. For actual commuting closure operators, a walk with crossing numbers (n, m) has holonomy $h^n (g^{-1})^m$, so equality of these numbers gives equality of the required transports. Inside an integer-lifted region, the crossing numbers are the differences of the endpoint integer deck coordinates. Simple connectedness of the actual cell domain derives the lift and hence makes those numbers independent of the interior path with fixed endpoints.³³

The exterior loop with the required numbers is now constructed from the actual simply connected cell domain. The full plane preimage of the region is the union of integer-period translates of its planar lift; distinct copies are disjoint even at sides and corners. The exterior collar of radius $3/4$ projects wholly outside the actual torus region. Each native crossing edge supplies an exterior integer center in that collar, and the difference of the two exterior deck

³⁰Results: `TNLean.PEPS.exists_positive_gram_regularOpenRegionMatrix_of_connected`, `TNLean.PEPS.exists_normalization_regularPhysicalCutState_of_connected`.

³¹Results: `TNLean.PEPS.regularTweezerMatrix_mem_unitaryGroup`, `TNLean.PEPS.regularTweezerEquiv_symm_apply_twisted`.

³²Result: `TNLean.PEPS.regularBoundaryTreeRoute_holonomy`.

³³Results: `TNLean.PEPS.regularRegionCycleWord_eval_conj`, `TNLean.PEPS.regularWalkHolonomy_torusClosure_eq_of_winding_eq`, `TNLean.PEPS.torusWalkWinding_eq_of_isSimplyConnected`.

coordinates is exactly the winding pair of the interior boundary route.

The collar paths are derived rather than assumed. Simple connectedness excludes finite four-neighbor components of missing integer cells. A finite box then shows that the entire missing-cell lattice is connected. The occupied-square contact graph is also connected, retaining shared corners. Orient each exposed cell edge and give the finite set of these edges its boundary-successor permutation. For a successor-invariant weight, the finite horizontal-ray potential is constant on missing cells and on occupied cells; its jump across an exposed edge is that edge's weight. Taking the indicator of one successor cycle shows that every exposed edge belongs to that cycle. Each successor transition yields an exterior four-neighbor walk supported by a common occupied square, and every nearby missing center joins such an exposed edge. Thus the safe exterior graph is connected. Its straight edge segments lie entirely in the genuine collar, giving the required continuous paths. Lattice-path approximation projects these paths to actual induced-complement walks with their exact deck-coordinate differences as crossing numbers. Appending the two complementary tree paths gives the fixed root loops for all boundary routes, independently of every closure label. No condition on group holonomies replaces the geometry.³⁴

The occupied connectivity condition must be distinguished from simple connectedness of the closed-cell realization. Two squares meeting only at a corner have a contractible union but no occupied four-neighbor path. The source asks for a contiguous block of spins (lines 1931–1933), and its blocking construction concerns an already connected block (lines 1349–1354). Accordingly, the contiguous-region statement uses connectivity of the actual induced occupied graph. This is the source's block condition; it is not inferred from the topological hypothesis. The contour and collar arguments themselves allow corner contacts.

The constructed loop word evaluates to $m_f m_{f_0}^{-1}$. Evaluating this word on z then gives $W_f(z) = x m_f m_{f_0}^{-1} x^{-1}$. The single controlled permutation $y_f \mapsto W_f(z)^{-1} y_f$, leaving the cycle coordinates fixed, turns $y_f = x m_f \theta_f$ into $x m_{f_0} \theta_f$ at every crossing bond. Its reversible coordinate permutation and unitary matrix on the native physical half-edge space are established, as is its exact action on the actual twisted canonical coefficient sum. The resulting common-transport formula at the reindexed boundary input is established. For fixed actual complementary loops with the prescribed geometric crossing numbers, the relative-word identity is now derived for every commuting closure pair. The resulting unitary is chosen before the closure labels; it depends only on complement coordinates and those fixed walks, whose existence follows from the geometric argument above. The common remaining translation is separated by the boundary reference coordinates in the actual canonical cut contraction. Its exact coefficient formula factors every coherent finite sector sum into the same normalized maximally entangled relative-boundary vector, the same normalized region ancillary vector, and a complementary cycle-orbit amplitude. All depen-

³⁴Results: `TNLean.PEPS.integerExteriorCollarGraph_connected_of_isSimplyConnected`, `TNLean.PEPS.exists_complementBoundaryLoops_of_isSimplyConnected`.

dence on the reference crossing multiplier cancels in the boundary sum. Region cycle residuals and relative-word identities are explicit premises of the finite-graph formula; the torus specialization derives them from the geometric lift and constructed complementary winding loops.³⁵

A canonical unitary commuting with the product of the local invariant projectors is proved to have a unitary implementation on the original physical region, without requiring surjectivity of the site maps. Its product Gram identity is derived from local isometry. For actual walk-word controls, the commutation identity is also proved: independent vertex translations act on the cycle labels by conjugation by the root translation, and evaluation of a free-group word respects that conjugation. Hence the controlled permutation commutes with each vertex translation and with their average, the actual product of local invariant projectors. It preserves the supported canonical physical space and has a unitary implementation on the original physical region.³⁶ The resulting common reduced matrix is now computed in the native canonical half-edge coordinates. For every coherent sum C , it satisfies $CC^\dagger = \text{tr}(CC^\dagger)\rho_R^0$, with ρ_R^0 fixed before the closure labels and coefficients. The actual original physical cut is the image of C under the two products of local site maps; their positive Gram factors and canonical support identities are derived from local isometry. Normalizing therefore gives one fixed density on the original physical region for every nonzero coherent commuting-closure sum. Its rank is $|G|^{b-1}$, its nonzero spectrum is flat, and every finite nonnegative Rényi entropy is $(b-1)\log|G|$. The positive-order calculation uses the genuine spectral trace of a real power, with the rank convention at zero and von Neumann entropy at one.³⁷ For a contiguous block with actually simply connected closed-cell realization, the induced complement is connected. Indeed, every complementary component has a crossing bond to the nonempty occupied region, and the collar argument joins all exterior crossing endpoints. The two spanning trees, their roots, the boundary numbering, and the winding loops can therefore be chosen within the proof. The common-density and entropy statements require none of these choices as hypotheses.³⁸

The regular-torus statements retain periods at least three. Periods one or two require a model retaining multiple bonds between the same sites and self-bonds; the simple-graph statements do not cover them. The operation on the two width-one coordinate-zero stripes is established for actual coherent superpositions of closure states whose closure pairs all commute: their complementary rectangle projects injectively from a compact convex planar rectangle, so the common-density theorem gives a physical complementary unitary relating

³⁵Results: `TNLean.PEPS.exists_unitary_torusComplementDisentangling_of_winding_eq`, `TNLean.PEPS.regularControlledProjectorCutMatrix_sum_factorization`, `TNLean.PEPS.exists_unitary_torusControlledBoundaryFactor_of_isSimplyConnected_and_winding`.

³⁶Results: `TNLean.PEPS.regularCycleControlledBoundaryMatrix_commute_localProjector`, `TNLean.PEPS.exists_unitary_regionPhysicalProductMatrix_mul_cycleWords`.

³⁷Results: `TNLean.PEPS.torusControlledCut_reducedMatrix_eq_trace_smul`, `TNLean.PEPS.IsGIsometric.exists_torusPhysicalCut_density_transfer`.

³⁸Result: `TNLean.PEPS.IsGIsometric.exists_torusPhysicalCut_common_density_of_isSimplyConnected`.

any two such normalized superpositions.³⁹ Identification with the entire two-dimensional parent-Hamiltonian ground space remains separate. This is the general-coordinate form of the controlled operation displayed in [SCPG10] at source lines 1961–1965, leading to the block decomposition displayed at lines 1970–1976 and the common reduced state for commuting closure superpositions.⁴⁰

8 Superpositions with complementary twists

The Gram calculation now includes actual group operators on internal bonds. For assignments u, w that are the identity on crossing bonds, suppose their common identity bonds contain a connected spanning subgraph of the region. The local translations are then constant. Let κ_R be the untwisted region factor and put

$$F_{u,w} = \{x \in G : u_e x = x w_e \text{ for every internal bond } e\}. \quad (16)$$

Then the actual cross-Gram operator satisfies

$$(T_R^u)^\dagger T_R^w = \frac{\kappa_R}{|G|} \sum_{x \in F_{u,w}} L_x^{\otimes b}, \quad (17)$$

$$P(T_R^u)^\dagger T_R^w P = \kappa_R \frac{|F_{u,w}|}{|G|} P. \quad (18)$$

The second identity uses only that every common translation acts as the identity on the invariant boundary.⁴¹

Consequently, let the original region D be connected, and consider any finite family of twists supported entirely inside D^c , with a common connected spanning subgraph of identity bonds in D^c . Every nonzero superposition of their actual PEPS contractions has the same normalized reduced density on D :

$$\rho_D = \frac{1}{c_D |G|^{b-1}} T_D P T_D^\dagger, \quad T_D^\dagger T_D = c_D P, \quad c_D > 0. \quad (19)$$

Its rank is $|G|^{b-1}$ and its nonzero spectrum is flat, so its von Neumann entropy is $(b-1) \log |G|$. The region and mixed Gram identities are derived from local regular isometry. Positivity of the combined complementary Gram factor follows from nonvanishing of the actual superposition, with no extra condition on its coefficients.⁴²

³⁹Result: `TNLean.PEPS.IsGIsometric.torusClosureSuperpositionCut_stripe_local_equality`.

⁴⁰Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 1935–1990 and 2027–2072.

⁴¹Result: `TNLean.PEPS.sum_regularBoundary_filtered_translation_eq_weightedTranslation`.

⁴²Result: `TNLean.PEPS.exists_normalization_regularTwistedPhysicalCutSuperposition`.

This removes the absence-of-twists restriction for this finite-cut setting. It still requires the twists to avoid every edge incident to D . The native torus closure states are now identified with these twisted finite-graph contractions. On a torus of width and height at least three, a positive-size coordinate rectangle strictly inside the fundamental domain has connected interior and connected complement even after removing the closure seams. The seams have no edge incident to the rectangle. Thus local regular isometry alone gives the same reduced density, rank, flat spectrum, and entropy for every nonzero superposition of native closures on such a rectangle. The boundary count and these connectivity conditions are proved from the coordinate bounds, rather than supplied as additional hypotheses. This restricted calculation permits noncommuting closure pairs as well.⁴³

Commuting closure seams may also be placed at arbitrary rows and columns: local virtual invariance removes the corresponding vertex changes of basis, while commutation preserves the transverse closure. This deformation identity is established for the actual torus contraction, without isometry or semi-regularity.⁴⁴ The geometry-derived common-density theorem covers contiguous actually simply connected regions meeting either closure seam, for commuting closure families on native tori of periods at least three.

For a regular G -injective tensor the original site map and a local inverse give two-way maps between the actual closure superposition and the regular averaging-projector superposition. These maps preserve Schmidt rank across the physical cut. The normalized reduced density therefore has rank $|G|^{\partial D - 1}$, and its genuine zero-order Rényi entropy is $(\partial D - 1) \log |G|$. This proves the regular native-torus version of Corollary 6.10 on the same contiguous simply connected regions.⁴⁵ Only rank is preserved by these local maps; flat spectrum is the additional isometric conclusion.

9 Scope: statements about the whole network

Two statements of the review concern the contracted lattice rather than one tensor.

1. The primal tensor's virtual symmetry under every irreducible representation π is formalized as the single-tensor identity

$$A_{t,r,b,l} \pi(t) \pi(r) \pi(b)^{-1} \pi(l)^{-1} = A_{t,r,b,l} \mathbb{I} \quad (20)$$

for every entry, which follows from $tr b^{-1} l^{-1} = 1$ on the support, and for the contracted network on the torus: the virtual string that multiplies $\pi(g)$ for each bond with label g it crosses can be pulled through every rectangle

⁴³Result: `TNLean.PEPS.exists_normalization_torusClosureSuperpositionCut_rectangle`.

⁴⁴Result: `TNLean.PEPS.torusBondNetwork_closureAt_eq_torusGClosure`.

⁴⁵Result: `TNLean.PEPS.IsGInjective.torusPhysicalCut_rank_zeroEntropy_of_isSimplyConnected`.

of sites, since on the support $\pi(t)\pi(r) = \pi(l)\pi(b)$ at each site and the two boundary paths of a grid of such squares have the same product. This item is resolved.

2. The review states that the equal-weight superposition of plaquette colorings is the equal-weight superposition of the configurations obeying Gauss' law. The primary source reads the contracted dual network as the superposition over all color patterns, with a Hamiltonian whose local term makes the four spins of each tensor multiply to the identity, whose plaquette term does the same for the spins of four tensors around a corner, and whose two-body term enforces equal weight.⁴⁶ On a plane, or on a disc with open boundary, every Gauss-law configuration is a difference of colorings. On a torus the differences of colorings are exactly the Gauss-law configurations of trivial holonomy around the two non-contractible cycles, so the statement holds for that sector only. This is formalized: with the dual tensor at every site of a torus of width and height at least three, the coefficient of the contracted network is $|G|$ at every spin configuration obeying the vertex and plaquette rules with trivial holonomy around one row and one column, and 0 elsewhere. The factor $|G|$ counts the colorings with given differences, which differ by one global shift. The proof writes a coloring through the left color f_v of each site; the spins then prescribe $f_{v+(1,0)} = a_v f_v$ and $f_{v+(0,1)} = b_v f_v$ for elements a_v, b_v built from the spins, and such f exist exactly when the a, b are flat around every elementary square, which is the plaquette rule, and have trivial holonomy around one row and one column. The review also prints the Gauss law as $g_1 g_2 g_3^{-1} g_4^{-1} = 0$, where the identity element 1 is meant.⁴⁷ This item is resolved within the trivial-holonomy sector, which is the whole state of the network.

10 Flux detection on a connected block

Fix a spanning tree of a connected regular block and one non-tree internal edge. Its exposed cycle coordinate is conjugate to the residual of the actual inserted group operators.⁴⁸ The diagonal projections onto its conjugacy classes commute with the product local averaging projector. Through the original local G -isometric maps A , they give physical projections $Q_C = c^{-1} A D_C A^\dagger$. An additional projection $I - \sum_C Q_C$ completes the measurement on the whole physical Hilbert space and annihilates every actual twisted block column. The outcome on a nonzero column is exactly the conjugacy class of the derived residual. This establishes the conjugacy calculation in the proof of Theorem 6.15 for a chosen cycle in an actual connected block. The same construction detects

⁴⁶Source traceability: `Papers/1001.3807/paper_v3.tex`, lines 2824–2827 and 2914–2923; `Papers/2011.12127/TN-Review-main.tex`, lines 2460–2461.

⁴⁷Source traceability: `Papers/2011.12127/TN-Review-main.tex`, line 2452.

⁴⁸Result: `TNLean.PEPS.regularProjectorTwistedRegionMatrix_coordinates`.

the conjugacy class of any specified closed-walk holonomy.⁴⁹ On native tori of periods at least three, the four-site plaquette walk and its inserted-bond holonomy give the corresponding physical plaquette measurement.⁵⁰ No parent-Hamiltonian assertion is used in these measurement results.

11 The semi-regular bond isometry

For each irreducible block, the map $X_i \mapsto m_i^{-1/2} X_i \otimes I_{m_i}$ preserves the Hilbert-Schmidt inner product. Taking the regular multiplicity $m_i = d_i$ carries the weighted block $\sqrt{d_i} D^i(gh^{-1})$ to $D^i(gh^{-1}) \otimes I_{d_i}$, exactly as in Section 7. The coefficient map and its product over any finite family of bonds are isometries, and the product map carries arbitrary coherent sums of the weighted bond products to their repeated-multiplicity counterparts.⁵¹ The identification of the fourth-root-weighted site tensor with those bond sums is a further statement; it is not assumed in this isometry calculation.

12 Four-block closure with independent coordinate spaces

The four literal boundary-cut ranges in [SCPG10, Theorem 5.5] now have their full source equality: their intersection is the span of the actual commuting closure networks.⁵² The eight virtual coordinate spaces and four physical spaces may all differ. Each bond carries its own matching semi-regular representation, each physical site is G -injective, and each boundary is one unrestricted joint tensor. Both inclusions are proved without a parent-kernel or supplied spanning hypothesis. The algebraic proof does not need unitarity, so the source's unitary convention is covered. This resolves the four-block closure statement; it does not replace the separate overlapping-region intersection or geometric renormalization statements.

13 Local regular parent constraints and the remaining closure step

For connected regions of regular G -injective tensors, local inverses expose the actual canonical regional coefficients. The original regional ground space has only identity cycle coordinates. A nonzero vector lying both in this space and in

⁴⁹Results: `TNLean.PEPS.exists_regularCycleClass_physicalMeasurement`, `TNLean.PEPS.exists_regularClosedWalkClass_physicalMeasurement`.

⁵⁰Results: `TNLean.PEPS.IsGIsometric.exists_torusPlaquette_physicalMeasurement`, `TNLean.PEPS.IsGIsometric.exists_torusPlaquette_fluxClass_measurement`.

⁵¹Results: `TNLean.PEPS.sum_trace_multiplicityBondFamilyMap_conjTranspose_mul`, `TNLean.PEPS.multiplicityBondFamilyMap_sqrt_smul`.

⁵²Result: `TNLean.PEPS.DependentTorus.fourCutSpace_eq_commutingClosureSpan`.

an inserted regional space therefore forces every internal closed-walk holonomy to be the identity. Simultaneous right multiplication of the two exposed half-edge labels of an internal bond also preserves these coefficients, including in the presence of arbitrary inserted left group multiplications. These are local coefficient consequences; no global representation is assumed.

On native tori with both periods at least three, the parent condition at the intersection of the two closure seams forces commuting closure labels. Conversely, commuting closures satisfy the original regional parents when the induced regions are connected and their actual closed-cell realizations are simply connected, including across either seam. A physical linear holonomy filter fixes the parent-supported space and discards every noncommuting closure. Thus, for a finite parent family containing the seam-intersection plaquette, let $\mathcal{C}_{\text{all}}$ be the span of all native closures and $\mathcal{C}_{\text{com}}$ the span of commuting closures. Under these regional hypotheses,

$$\ker H \cap \mathcal{C}_{\text{all}} = \mathcal{C}_{\text{com}}. \quad (21)$$

The dimension of this intersection is the number of commuting pair-conjugacy classes. No G -isometry is required, and the linear filter is not asserted to be Hermitian.

For the complete native plaquette family, the arbitrary-kernel spanning step is now proved for regular virtual representations. Apply local left inverses to a physical parent ground vector. The resulting half-edge function is invariant under vertex left translations and simultaneous right translations on each bond, and vanishes outside flat plaquette connections. Its explicit finite orbit expansion contains only flat insertions. Applying the original site maps reconstructs the physical vector, and a tree gauge reduces each flat insertion to two commuting seam holonomies. Thus

$$\ker H = \mathcal{C}_{\text{com}}, \quad \dim \ker H = \#(\{(g, h) : gh = hg\}/G), \quad (22)$$

where the quotient is by simultaneous conjugation. The reverse inclusion uses the connected, simply connected realization of every native plaquette; translation includes plaquettes crossing either seam. The interactions may be any positive operators with exactly the regional PEPS spaces as their kernels. No isometry or prior closure expansion is assumed.

This proves the regular-representation, native-torus specialization of [SCPG10, Theorems 5.7 and 5.9] when both periods are at least three. The same full-kernel equality and dimension formula now also hold for any uniform unitary semi-regular virtual representation and any one G -injective site tensor repeated at every vertex.⁵³ The representation blocks and their actual multiplicities are derived. Exact open-boundary transport identifies the genuine regional parent spaces; edge-covering support supplies both inverses of the whole-parent equivalence. An explicit native-orientation bridge gives the regular parent dimension.

⁵³Results: `TNLean.PEPS.IsGInjective.torusParentKernel_eq_commutingClosureSpan_of_isSemiRegular`; `TNLean.PEPS.IsGInjective.finrank_torusParentKernel_of_isSemiRegular`.

Separately, moving commuting seams beyond each plaquette proves membership of the actual closure vectors using local invariance alone. Their established independence and the matching dimension prove that they exhaust the kernel. No local-range identity or ground-state expansion is assumed.

The source’s link-dependent representations and arbitrary blocked site tensors remain outside this uniform native capstone. Smaller-period models also require a convention retaining parallel bonds and loops. These remaining source obligations are tracked in issue #8675.

14 Finite torus flux operations

The flux detection, motion and creation results of [SCPG10, Theorems 6.15–6.17] are formulated in the bulk, with remote boundaries; source lines 2174–2342. The formalized regional identities retain every actual crossing label and are uniform in those labels. Their original-spin operators are obtained from the local regular G -isometries, rather than supplied as hypotheses.

The finite simple-graph torus requires both periods to be at least three for the single-plaquette detector. The translated two-plaquette construction uses horizontal period at least four and vertical period at least three. The three-plaquette strip uses horizontal period at least five and vertical period at least three. All positions, including both periodic seams, are allowed. These period bounds exclude extra induced bonds in the selected blocks. Thus the corresponding statements are finite-geometry specializations, rather than formalizations of an unrestricted lattice statement. Extending the lattice model to smaller periods is separate work. The stated contraction identities make no assertion about excitation energies or the entire parent kernel.

15 The selected chronological physical braid

The braiding passage of [SCPG10, §6.6], source lines 2343–2411 and equation “eq:anyons:fluxon-braiding-lazy”, conjugates one flux and detects the commutator class after reunion. The finite realization uses a specified twelve-step route on a native torus with horizontal period at least eight and vertical period at least seven. Each step is a unitary on the original spins; the gauges, residuals and empty neighboring plaquettes are derived from the actual inserted bond assignment. Their chronological product acts on the genuine closed contraction. A complete six-spin projection-valued measurement, chosen before the two flux labels, detects the commutator class on the resulting physical cut. The based transport at reunion is conjugate to the commutator, which leaves this measured class unchanged.

The result covers this route at every translated position, including both seams. Other braid representatives, smaller periods and the irreversible merging operation require further arguments.

16 Native charge detection and correlated pair readout

Electric charges in [SCPG10, §6.6], source lines 2432–2558, are diagonal character insertions. The two-site detector is a complete projection-valued measurement on the original spins, chosen before the irreducible character, internal parameter and boundary labels. The formalization derives its action from the actual open contraction on every native horizontal or vertical edge of tori with both periods at least three, including both seams. Local regular G -isometry also gives a nonzero local boundary column. This does not assert that a lone charge has nonzero contraction on a fully periodic vacuum: total-charge constraints and the source’s remote boundaries must be distinguished from a local-column statement.

The correlated pair has coefficient $\chi(ph^{-1}g)$, as in `eq:anyons:chargeon-pair-state`. On the translated six-site native block, horizontal period at least three and vertical period at least four suffice for one physical unitary to create this insertion before every boundary configuration. Two complete two-spin measurements, fixed before χ and p , then identify χ and its contragredient at the two endpoints of the same prepared state. The action is computed inside the actual retained-label sum. Neither a column factorization nor a prescribed sector action is assumed. The finite-geometry statement is explicit; its extension to unrestricted lattice geometries remains separate work. The finite-torus charge-motion result below also factors into independent unitaries on the two actual two-spin columns.

17 A selected-block charge parameter operation

Let R be a finite block with a chosen spanning tree and root, and choose an internal charge bond e and a non-tree flux bond b . Suppose the original sites are locally G -isometric for the regular incident-edge action. Retain the literal diagonal weight $\chi(p\eta_e)$ in the actual open contraction, where η_e is the tail bond label. Denote this vector by $\Psi_R(u; \chi, p, \theta)$, with actual group operators u and boundary configuration θ .

There is one unitary $W_{e,b}$ on the original spins of R , chosen before u , the coefficient function χ , $p \in G$ and θ , such that

$$W_{e,b}\Psi_R(u; \chi, p, \theta) = \Psi_R(u; \chi, pK_{u,e,b}^{-1}, \theta), \quad K_{u,e,b} = k_{\text{tail}(e)}\rho_b k_{\text{tail}(e)}^{-1}.$$

Here k is the actual rooted tree gauge and ρ_b its cycle residual; both are derived from u . The character label is retained when χ is an irreducible character. The weighted contraction is computed directly: diagonal character insertions are not inferred from linear combinations of group-representation matrices.

The construction realizes the charge parameter change of [SCPG10, §6.6], source lines 2569–2581, on this selected finite block. Its accessible coordinates come from source lines 1765–1920. Relating this operation to the complete

geometric charge–flux braid, with moving endpoints, requires further arguments. These identities concern contracted vectors and local unitaries.

Original-spin charge motion. For the regular action, the diagonal charge insertion on the upper horizontal bond of an actual four-site plaquette can be moved to its lower horizontal bond by one unitary on the original four spins. The unitary is selected before the character, internal parameter and every actual boundary configuration; these data are retained. The proof derives the tensor contraction identity from the internal-reference exchange, and chooses all tree data from the actual induced connectivity. Translated plaquettes crossing either periodic seam are included, with both torus periods at least three. This proves the finite-torus movement diagram of [SCPG10], `eq:anyons:chargeon-move-setting`, lines 2489–2507. The source proof factors its operation into independent unitaries on the two columns. This factorization is now derived on the actual torus plaquette: each connected two-spin column admits its own boundary-register exchange, and the original open contractions join by summing each common bond label exactly once. The literal Kronecker product of the two physical unitaries moves the diagonal insertion before every character, parameter and exterior label. No joining formula or local Gram identity is supplied as a hypothesis. Extension to unrestricted lattice geometry remains separate.

Weighted charge–flux string deformation. Vertex gauge changes preserve the actual bond contraction with arbitrary scalar diagonal weights. The open-region identity retains correlated weights and explicitly transports the actual boundary labels. On the native translated string figure, the central charge diagonal $\chi(pt)$ becomes $\chi(pk^{-1}t)$ in either displayed swept presentation, including both periodic seams, with torus periods at least five and four respectively. For a correlated pair, the transformed coefficient is $\chi(p\eta_1^{-1}k^{-1}\eta_0)$ when only the first selected tail is swept. The flux factor is retained at its literal position between the pair labels.

This proves an equality of actual original-spin contractions in different gauge presentations, following [SCPG10], lines 2560–2581. The literal flux string now determines an original-spin operation on an actual translated eight-site strip, with periods at least five and four. Its spanning-tree gauge is derived from the seven native transports: it is trivial on the lower row and equal to the flux on the upper row. The endpoint cycle residual determines whether the physical unitary or its adjoint is used at a seam. One unitary precedes every flux, coefficient function, charge parameter and boundary configuration. It implements $\chi(p\eta_0) \mapsto \chi(pk^{-1}\eta_0)$ and, uniformly in an unsummed partner label, $\chi(p\eta_1^{-1}\eta_0) \mapsto \chi(p\eta_1^{-1}k^{-1}\eta_0)$. The partner bond below the strip is derived to lie outside. Resolving its actual label and summing the native cut identity gives the same operation on the genuinely contracted correlated closed state. No factorization or global state action is supplied. The remaining step is to identify the complete prescribed geometric charge–flux route and the normalized prepared-

pair reunion with these actual coefficients; neither a gauge-presentation equality nor this auxiliary local crossing alone establishes the full braid of lines 2560–2615.

18 Closed pair preparation

The original-spin charge-pair preparation is now connected to the actual closed correlated contraction on a finite graph. For a chosen connected region with a spanning tree and two distinct internal bonds, local regular G -isometry derives one regional unitary preparing the literal weight $\chi(p\eta_{e_1}^{-1}\eta_{e_0})$, uniformly in the crossing labels. Contracting the unchanged exterior gives the corresponding closed state; it is nonzero and has precisely the norm of the unmodified closed state. This auxiliary preparation result does not yet identify preparation in the prescribed flux-string background or the complete geometric braid and reunion of [SCPG10], lines 2560–2615.

19 Accessible-register interference

For a unitary irreducible character χ of dimension d , the normalized initial pair and its literal flux-inserted coefficient vector are

$$v(g, h) = |G|^{-1}\chi(ph^{-1}g), \quad v_k^x(g, h) = |G|^{-1}\chi(ph^{-1}xk^{-1}x^{-1}g).$$

Translated character orthogonality gives

$$\|v_k^x\| = \|v\| = 1, \quad \langle v, v_k^x \rangle = \chi(k^{-1})/d,$$

independently of p and x . The complete binary measurement $D = |v\rangle\langle v|, 1 - D$ therefore returns the initial pair with probability $|\chi(k)/d|^2$, as in [SCPG10], source lines 2582–2615.

These are exact statements about the two accessible group registers. The measurement on the six original physical sites and its normalized input from the complete geometric charge–flux braid require further contraction and physical-operation identities. They are not inferred from a prescribed decomposition of the global state.

Return projection in the actual finite-region contraction. For a finite block with a chosen tree and two distinct internal bonds, the extended reference return projector now acts inside the literal correlated canonical bond sum. The unknown common translation and every original boundary label are retained; the accepted column is the initial column multiplied by $\chi(k^{-1})/d$. Hermitian idempotence is derived through the actual reference-coordinate extension. This establishes the canonical projection step of [SCPG10], source lines 2582–2615. Transport to the original physical spins and its native six-site specialization are separate results; no complete geometric braid is asserted by this canonical statement.

Original-spin return operation. The return projection is now a complete positive binary measurement on the original physical spins of a finite regularly G -isometric block. It is chosen before the flux and all actual boundary labels; its accepted literal correlated column is the initial column multiplied by $\chi(k^{-1})/d$. The rejected column is the difference. The actual translated six-site torus patch supplies its geometry internally, with horizontal period at least three and vertical period at least four, including both seams. This establishes the local original-spin return operation in [SCPG10], lines 2582–2615, without a given decomposition of the physical state. Identification of a complete prescribed braid with this input and a global normalized probability remains separate from the present local theorem.

20 The unblocked toric-code tensor

The four elementary binary tensors of `eq:ex:kitaev-tens` are contracted around the actual checkerboard square, retaining all eight boundary legs and all four original physical spins. After a CNOT on each doubled boundary pair, the tensor map factors as

$$BC = KE^\dagger.$$

Here C is the unitary boundary permutation, E includes four color registers with four auxiliary bits fixed at zero, and K is the existing dual toric-code tensor with the explicit cyclic physical reordering. The coefficient identity follows the construction in [SCPG10], source lines 2755–2827.

For torus periods at least three, the actual open contraction of any translated four-site plaquette, with its four corner orientations prescribed as in the checkerboard diagram, is identified with this blocked tensor for every native boundary assignment and original spin configuration. All eight crossing bonds are enumerated bijectively, including blocks crossing either seam. Conjugating the pairwise boundary CNOT by this enumeration gives a unitary native boundary transformation and the exact identity

$$A_v C_v = K_v E_v^\dagger,$$

where E_v is an isometric inclusion of the four retained boundary bits. These local tensor-map identities now have a global counterpart for the untwisted binary checkerboard state on a torus of even periods $2w, 2h$, with $w, h > 0$. Division into 2×2 squares gives a disjoint tiling compatible with every nearest-neighbor bond, including periodic seams. Reindexing all fine bonds into internal and crossing labels and then summing the internal labels gives the product of the actual blocked tensors. Every surviving crossing pair has equal labels, so the global contraction is exactly the color-difference PEPS on the $w \times h$ torus, with no extra scalar. A bijective physical regrouping preserves inner products in both directions and sends the complete fine state to this coarse state. The redundant registers eliminated in this calculation are virtual.

Only identity bond matrices occur: no twisted torus sector is asserted. The physical map is regrouping, not an independently constructed physical CNOT

disentangler. Odd fine periods are outside this tiling. The arbitrary regular- G physical reduction to the same coarse tensor and neighboring maximally entangled pairs in Observations 6.5–6.6 remains a separate claim.

21 Observables of arbitrary parent ground vectors

Within the uniform regular native-torus setting with both periods at least three, the full positive parent kernel now supplies the finite commuting closure coefficients for every ground vector. The direct kernel statements therefore give common reduced density, boundary rank, flat spectrum, von Neumann entropy and all finite real-order Rényi entropies with $\alpha \geq 0$ for each nonzero ground vector on connected regions whose closed-cell realization is simply connected. They also give equal local expectations and complementary unitaries, including the two width-one stripes. Regular G -injective tensors retain the rank and zero-order entropy conclusions only; flatness is not transported through arbitrary inverses. The relevant declarations include `TNLean.PEPS.IsGInjective.exists_torusGClosure_sum_of_mem_parentKernel`, `TNLean.PEPS.IsGIsometric.exists_torusParentKernel_common_density_of_isSimplyConnected`, and `TNLean.PEPS.IsGInjective.torusParentKernel_rank_zeroEntropy_of_isSimplyConnected`. These are corollaries of the full regular parent classification and existing cut results. They do not enlarge the region, period, tensor or entropy-order hypotheses.

22 Independent-dimensional native parent classification

For native torus periods at least three, each edge may now carry its own finite-dimensional semi-regular representation and each site its own G -injective tensor. The complete positive microscopic plaquette-parent kernel equals the actual commuting-closure span, and its dimension is the number of simultaneous conjugacy classes of commuting pairs: `TNLean.PEPS.ker_torusPlaquetteParentHamiltonian_eq_nativeCommutingClosureSpan` and `TNLean.PEPS.finrank_ker_torusPlaquetteParentHamiltonian`. Independent finite physical alphabets are faithfully zero-extended into the existing parent framework. Every ambient kernel vector is reconstructed by product restriction; no local-image assumption is imposed. `TNLean.PEPS.nonuniformNativeParentEquiv` is the explicit original-closure-span equivalence, with dimension given by `TNLean.PEPS.finrank_ker_nonuniformTorusPlaquetteParentHamiltonian`. The proof derives a coherent bond expansion, flatness and seam holonomies from microscopic parent membership. It does not assume a four-cut or closure expansion. Periods one and two still need a labelled incidence model with parallel bonds or loops; that exceptional geometry is not covered by a simple-graph theorem.

For the binary Kitaev checkerboard tensor (four colors on one translated four-site block) the local physical support reduction is also derived; it is not claimed for the other blocks of the classification above. A full four-spin parity permutation extracts one zero physical bit. If $K_3(s, p)$ is the indicator of the three adjacent color differences, the explicit color-coordinate bijection gives $K_3 K_3^\dagger = 2I$. Thus $J = 2^{-1/2} K_3^\dagger$ is an isometry. With $E_{P,v}$ and $E_{V,v}$ the physical and virtual zero-register inclusions, the normalized actual transformed tensor satisfies

$$2^{-1/2} R_v A_v C_v = E_{P,v} (E_{V,v} J)^\dagger. \quad (23)$$

Its initial and final support projections follow from (23). This supplies the local physical implementation on the support described in the footnote of [SCPG10], source lines 2772–2776. Global tiling remains separate.

23 Verdict and elimination plan

The note is a scope restriction: the review’s identification of the contracted dual network with the equal-weight superposition of all Gauss-law configurations holds on the torus for the trivial-holonomy sector, which is the sector formalized; the configurations of nontrivial holonomy have coefficient 0. The normalization and the representation are local corrections within that restriction and are conventions that need no further work. The pulling-through identity of the representation operators across the contracted primal network is formalized on the torus. For these torus statements no elimination is planned: the unrestricted reading is false on the torus.

Stability of G -isometry under concatenation is formalized in one dimension and for the contraction of one link in two dimensions. The connected-region Gram calculation extends the untwisted result to multiple bonds and closed cycles. The torus closure overlaps and independence of their regular isometric class vectors are established. For contiguous regions with actually simply connected closed-cell realization on tori of periods at least three, every nonzero coherent superposition of commuting closures has the same physical reduced density, flat rank $|G|^{\partial D - 1}$, and every finite nonnegative Rényi entropy $(\partial D - 1) \log |G|$. Cuts may cross either closure seam; the exterior loops and accessible-coordinate factorization are derived. The union of the two coordinate-zero width-one stripes supports a physical unitary relating any two normalized coherent closure states. For regular G -injective tensors, two-way local maps give the same boundary rank and zero-order Rényi entropy, without an isometric hypothesis. The remaining restrictions include the smaller-period lattice model and the stated observable/region hypotheses. Link-dependent parent classification and nonuniform physical tensors are now covered as described above. For any uniform unitary semi-regular representation, one repeated G -injective tensor and periods at least three, the full parent kernel is identified with the actual commuting-closure span. These and the other source results are tracked in issue #8271.

References

- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.
- [SCPG10] Norbert Schuch, J. Ignacio Cirac, and David Pérez-García. PEPS as ground states: Degeneracy and topology. *Annals of Physics*, 325(10):2153–2192, 2010.