

Two-Dimensional PEPS Examples: Tori of Width Two

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Verdict: scope-restriction (open).

1 What the sources print

Appendix A of [CPGSV21], under “Two dimensions: PEPS”, gives single-site tensors with four virtual indices ordered top, right, down and left, to be placed at every site of a square lattice.¹ The cluster state is written with the tensor $A = |0\rangle\langle 00++| + |1\rangle\langle 11--|$ on the square lattice, with no restriction on its size.² The CZX state of [CLW11] is placed on a torus of $2N \times 2M$ qubits, blocked into $N \times M$ sites of four qubits, and is the product over plaquettes of the states $|0000\rangle + |1111\rangle$; its tensor is $\sum_{i,j,k,l} |ijkl\rangle\langle (i,j), (j,k), (k,l), (l,i)|$ of bond dimension four.³ The AKLT state places a singlet on every link of the square lattice and projects the four spin- $\frac{1}{2}$ at each site onto their symmetric subspace, and the nearest-neighbour RVB state is the superposition of all coverings of the lattice by nearest-neighbour singlets; the review writes both as PEPS with tensors of the form $A = P(\mathcal{K} \otimes \mathcal{K} \otimes Y \otimes Y)$, again with no restriction on the lattice size.⁴ A classical model with nearest-neighbour interaction h at inverse temperature β gives the tensor $A = [\sum_i |i\rangle\langle i i i i|](\mathcal{K} \otimes \mathcal{K} \otimes M \otimes M)$ with $M = \sum_{i,j} e^{-\beta h(i,j)/2} |i\rangle\langle j|$, on the square lattice with no restriction on its size.⁵ All five constructions are local, so they apply for every $N, M \geq 1$ once the lattice is read as a multigraph: a site on a torus of width two has two distinct horizontal bonds to its neighbour, one on each side.

2 The restriction

The formal PEPS lives on a finite simple graph, and the torus is the simple graph on $\mathbb{Z}_w \times \mathbb{Z}_h$ with cyclic nearest-neighbour edges. For $w = 2$ the right and

¹Source traceability: `Papers/2011.12127/TN-Review-main.tex`, line 2415.

²Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2424–2429.

³Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2502–2514; `Refere`
`nces/1106.4752/source/dSPTmodel.tex`, lines 317–321.

⁴Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2432–2448.

⁵Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2483–2494.

the left neighbour of a site coincide, and so do the two edges joining them: the four legs of a site are not four distinct bonds. For each of the five constructions, the contraction on this graph is therefore a different network from the source's at $w = 2$ or $h = 2$. For the RVB tensor the difference is visible on the coefficients: at $w = 2$ the right and left legs of a site carry the same bond label, so a horizontal dimer would occupy two legs of the same site and the tensor vanishes on it. On a torus of 2×3 sites the formal network then has only vertical dimers, which cannot cover a cycle of three sites, and its state is zero, while the source's state, with its two horizontal bonds between the two sites of a row, is a nonzero superposition of horizontal coverings. For $w, h \geq 3$ the right and up edges of the sites are pairwise distinct and exhaust the edge set, and the formal statements hold there:

1. the cluster tensor on a torus of $N = wh$ sites has coefficient $2^{-N}(-1)^{\sum_{\langle u,v \rangle} \sigma_u \sigma_v}$, the sum running over nearest-neighbour bonds;
2. the CZX tensor has coefficient 1 at a configuration in which the four qubits around every plaquette agree, and 0 otherwise;
3. the AKLT tensor gives the symmetric projector at every site applied to a singlet on every edge;
4. the RVB tensor gives the sum over nearest-neighbour dimer coverings of the torus of the product of the singlets on the covered edges;
5. the classical-model tensor of a nearest-neighbour interaction h at inverse temperature β has coefficient $e^{-\beta H(\sigma)/2}$, where H sums h over the nearest-neighbour bonds, so its diagonal expectation values are those of the Gibbs state $e^{-\beta H}/Z$.

The GHZ example does not need the additional restriction $w, h \geq 3$: its coefficient depends only on whether the configuration is constant, and it is proved on every torus with $w, h \geq 2$. It still excludes a width or height of one, because a simple graph has no self-loops. The direct contraction with separate horizontal and vertical bond arrays is nevertheless proved to be the GHZ state for every positive w, h , including self-loops and parallel bonds (`TNLean.PEPS.sum_prod_ghzSiteTensor`). Its on-site permutation symmetry and the non-injectivity of its site tensor are also proved without a lower bound of two on the lattice sizes. Thus the GHZ statement itself has no remaining size gap; only its simple-graph representation excludes self-loops.⁶

3 Elimination plan

Formalize PEPS on finite multigraphs, where a torus of width two has two edges between neighbouring sites, and restate the five examples there. The

⁶Results: `TNLean.PEPS.stateCoeff_clusterPEPS`, `TNLean.PEPS.stateCoeff_czxPEPS`, `TNLean.PEPS.stateCoeff_akltPEPS`, `TNLean.PEPS.stateCoeff_rvbPEPS`, `TNLean.PEPS.stateCoeff_classicalPEPS`, `TNLean.PEPS.stateCoeff_ghzPEPS`.

proofs carry over unchanged once the edges of the torus are parametrized by the right and up edges of the sites.

References

- [CLW11] Xie Chen, Zheng-Xin Liu, and Xiao-Gang Wen. 2D symmetry protected topological orders and their protected gapless edge excitations. *Phys. Rev. B*, 84:235141, 2011.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.