

The CZX PEPS: Actual Rectangle Boundaries and Their Symmetry

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Verdict: scope-restriction (open).

1 What the source states

The on-site $\mathbb{Z}_2$ symmetry of the CZX model is generated by $U_{CZX} = U_X U_{CZ}$, with $U_X = X_1 X_2 X_3 X_4$ and $U_{CZ} = CZ_{12} CZ_{23} CZ_{34} CZ_{41}$ on the four qubits of a site, and the plaquette ground state is invariant under it [CLW11].¹ On a region with boundary, the two qubits of a boundary half plaquette are constrained to the span of $|00\rangle$ and $|11\rangle$ and form one effective spin, and projecting the symmetry of the region onto the support of its reduced state gives the effective boundary symmetry $\prod_i \tilde{X}_i \prod_i \widetilde{CZ}_{i,i+1}$, a matrix product unitary of bond dimension two.² The review states that the CZX PEPS of bond dimension four is invariant under this on-site symmetry and that its virtual action is the CZX matrix product unitary [CPGSV21].³

2 What is formalized

With the bond orientation of the note on the CZX tensor, each virtual leg carries two qubits (a, b) , and the following hold.

1. The CZX PEPS on a torus of width and height at least three is invariant under U_{CZX} applied at every site.
2. For one site tensor A , applying U_{CZX} to the physical index equals applying $V = (X \otimes X) CZ$, that is $V|ab\rangle = (-1)^{ab}|\bar{a}\bar{b}\rangle$, to each of the four virtual legs.

¹Source traceability: `References/1106.4752/source/dDSPTmodel.tex`, lines 283–312.

²Source traceability: `References/1106.4752/source/dDSPTmodel.tex`, lines 330–345 and 377–385.

³Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2502–2516 and 2582–2587.

3. Let a closed chain of N legs carry the pairs (c_m, c_{m+1}) , indices modulo N , so that neighbouring legs carry the same qubit of their common plaquette, and let ι be the resulting isometry from N effective spins into the legs. Then $V^{\otimes N} \iota = \iota U_N$ with $U_N = X^{\otimes N} D_N$ and $U_N |c\rangle = (-1)^{\sum_m c_m c_{m+1}} |\bar{c}\rangle$, and $\iota^\dagger V^{\otimes N} \iota = U_N$.

The review's two printed local matrices instead contract to

$$O(A) = D_N X^{\otimes N} = (-1)^N U_N.$$

Its bra-ket display with normalized $|\pm\rangle$ has an additional factor $2^{-N/2}$ and is not the unitary normalization. These order and normalization distinctions are recorded in `mpu_czx_tensor_normalization.tex`. The rectangle perimeter P is even, so the printed-matrix operator agrees exactly with U_P in the physical rectangle theorem below.

3 The actual rectangle map

The rectangular-region part of the former restriction is now proved for $R = [x_0, x_0 + w) \times [y_0, y_0 + h)$ in a $W \times H$ torus, with $W, H \geq 3$, $w, h > 0$, $w < W$, $h < H$, $x_0 + w \leq W$, and $y_0 + h \leq H$. Rectangles may meet a coordinate seam. Write $P = 2w + 2h$.

Let A_R denote the actual open-region contraction: only incident bond labels occur, internal bonds are summed, and crossing bonds are fixed. The clockwise enumeration of the P crossing bonds reverses bottom and left pairs. A nonzero physical coefficient forces those oriented labels to have the form (c_i, c_{i+1}) , with the successor taken modulo P . This follows from actual shared internal bonds along each side and from one site tensor at each corner. It is not a hypothesis on the region map.

Define $F_R |c\rangle = A_R |\mu(c)\rangle$, where $\mu(c)$ is the corresponding native boundary assignment. Conversely, arbitrary bits c on the distinct perimeter plaquettes extend to torus plaquette bits. Copying them into the four physical qubits at each site gives a configuration σ_c for which $F_R(\sigma_c, c') = \delta_{c, c'}$. Thus F_R is injective and its columns span exactly the image of A_R , of dimension 2^P .

Applying the one-site pull-through identity inside the actual contraction cancels every internal controlled phase twice. The remaining factors are on the actual crossing bonds. In the effective coordinates this gives

$$U_R F_R |c\rangle = (-1)^{\sum_i c_i c_{i+1}} F_R |\bar{c}\rangle.$$

Pair reversal preserves its controlled phase and commutes with simultaneous bit flip. The identification is therefore compatible with the source's physical on-site symmetry and with the native bottom/left bond convention. Since P is even, it also gives $U_R F_R = F_R O(A)$ for the review's printed matrix product operator, without a remaining sign.

4 Physical reduced-density support

The same plaquette witnesses identify the support of the actual regional reduced density. Fix effective boundary bits c and extend them to global plaquette bits. On the complement, the copied physical configuration has open coefficient one at the selected native crossing labels and zero at all other labels. The actual closed-state contraction across the cut therefore contains the column $F_R|c\rangle$ as a literal physical slice. Using the coefficient-matrix factorization $\rho_R = MM^\dagger$ gives

$$\text{ran } \rho_R = \text{im } F_R, \quad \text{rank } \rho_R = 2^P.$$

Here ρ_R is the unnormalized partial trace of the actual closed PEPS. Its trace is nonzero because it is positive semidefinite of positive rank. Division by that trace preserves the proved support and rank.

5 Remaining scope

The exact support and its dimension, including trace normalization, are now proved for the stated rectangles. The source's explicit reduced-state projection, its nonzero eigenvalues, the Gram normalization of F_R , and entropy remain open. The support theorem alone does not imply flatness or that F_R itself is an isometry. General nonrectangular regions and parent-Hamiltonian ground-space completeness are not asserted. The torus-size and bond-orientation conventions in the separate notes are unchanged.

6 Formal statements

1. `TNLean.PEPS.czxCRectangleBoundaryLabels_mem_range_of_openRegionWeight_ne_zero`.
2. `TNLean.PEPS.exists_czxCRectangleBoundaryMatrix_selector`, `TNLean.PEPS.czxCRectangleBoundaryMatrix_mulVec_injective`, `TNLean.PEPS.range_openRegionMap_czxCRectangle`, `TNLean.PEPS.finrank_range_openRegionMap_czxCRectangle`.
3. `TNLean.PEPS.regionPhysicalMap_czxOnSite_openRegionWeight`, `TNLean.PEPS.regionPhysicalMap_czxCRectangleBoundaryMatrix_column`, `TNLean.PEPS.regionPhysicalProductMatrix_czxCRectangleBoundaryMatrix_review`.
4. `TNLean.PEPS.range_regionReducedDensity_czxCRectangle`, `TNLean.PEPS.rank_regionReducedDensity_czxCRectangle`, `TNLean.PEPS.trace_regionReducedDensity_czxCRectangle_ne_zero`, `TNLean.PEPS.range_normalizedRegionReducedDensity_czxCRectangle`, `TNLean.PEPS.rank_normalizedRegionReducedDensity_czxCRectangle`.

References

- [CLW11] Xie Chen, Zheng-Xin Liu, and Xiao-Gang Wen. 2D symmetry protected topological orders and their protected gapless edge excitations. *Phys. Rev. B*, 84:235141, 2011.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.