Paper Comparison for BNT Inputs in the Non-periodic MPS Fundamental Theorem

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Verdict: open-gap (open).

1 Source passages and mathematical content

The non-periodic matrix product vector Fundamental Theorem in [CPGSV16, CPGSV17, CPGSV21] is a theorem about tensors already put in canonical form. The comparison is not stated as a theorem about arbitrary block-diagonal data plus unrelated finite-length hypotheses. The finite-length hypotheses now isolated in the formalization are not separate assumptions in the paper theorem. They are names for the mathematical ingredients supplied by the source argument: canonical-form reduction, BNT selection, fixed-length block-injective span after blocking, phase-gauge matching, and power-sum comparison.

This note uses the paper source, not issue numbers, as the comparison standard. The formal issue trackers are useful for scheduling work, but the mathematical boundary is the following one. A conditional formal theorem matches the source only when its extra hypotheses are exactly one of the source ingredients listed below, or when the text states that the source ingredient has not yet been proved.

Positive-length MPVs. The paper works with

$$\langle i_1, \ldots, i_N | V^{(N)}(A)\rangle = \operatorname{tr}(A^{i_1} \cdots A^{i_N}), \qquad N = 1, 2, \ldots.$$

A formal theorem should not use the empty word as a CPSV hypothesis or conclusion.

Canonical form. The source first reaches

$$A^i = \bigoplus_k \mu_k A_k^i, \qquad \sum_k D_k \le D,$$

and then removes periods by blocking. The formal route should therefore keep zero blocks and period-removal data separate from later injectivity blocking.

BNT decomposition. The source writes

$$A^i = X \left[\bigoplus_j (M_j \otimes A_j^i) \right] X^{-1},$$

and

$$|V^{(N)}(A)\rangle = \sum_j \left(\sum_q \mu_{j,q}^N \right) |V^{(N)}(A_j)\rangle.$$

The comparison is between the selected BNT representatives, not the raw cyclic-sector or repeated-copy lists produced on the way to them.

Block-injective span. After the cited blocking step, every matrix in

$$\bigoplus_j M_{D_j}(\mathbb{C})$$

is a linear combination of the block-evaluations $\tilde{A}^i$. The formal proof must prove this homogeneous fixed-length direct-sum span, or keep it explicitly as a hypothesis.

Equal-case weights. Once the BNT representatives have been matched, the source compares

$$\sum_q \mu_{j,q}^N = \sum_q (\nu_{j,q} e^{i\phi_j})^N$$

for all matched j and N, and only then applies the finite power-sum lemma.

1.1 Verdict

There is no mathematical discrepancy with the source theorem if the conditional formal hypotheses are read as names for source inputs that still have to be produced from the canonical-form proof. The source theorem assumes two tensors already in canonical form and then compares their bases of normal tensors. It does not assume arbitrary phase covers, raw cyclic-sector lists, cumulative pair-span fullness, or length-zero matrix product vector equality.

The present formal boundary is therefore a proof gap, not a different theorem. The remaining inputs are the representative-family BNT cover, the exact-length direct-sum span supplied by block-injective canonical form, and the transport of those data through the common blocking and word-reindexing steps. Until these inputs are proved, the downstream after-blocking theorem should stay visibly conditional.

The phrase "zero tail" is formalization terminology. The 2016 source says only that, after extracting the nonzero invariant blocks, $\sum_k D_k \leq D$, and explains this as the possible presence of zero blocks [CPGSV16, lines 214–219]. Thus a zero tail in the formal text means the unused all-zero block summand of dimension $D - \sum_k D_k$. It is not a paper term and should not be used without this definition.

1.2 Source label map

The comparison below uses the labels in the local 2016 source wherever the paper gives one:

$$A^i = \sum_{k=1}^{r} P_k B^i P_k = \bigoplus_{k=1}^{r} \mu_k A_k^i \qquad (\texttt{eq:II\_Aiplusk1})$$

is the canonical-form reduction; the following source sentence adds $\sum_k D_k \leq D$, explaining the possible all-zero leftover block. The canonical-form display is the equation labelled `eq:II_CF1`:

$$A^i = \bigoplus_{k=1}^{r} \mu_k A_k^i \qquad (\texttt{eq:II\_CF1})$$

The phase-gauge relation used in the BNT characterization is labelled `eq:II:A=XAX`:

$$\tilde{A}_k = e^{i\phi_k} X_k A_j X_k^{-1} \qquad (\texttt{eq:II:A=XAX})$$

The BNT characterization is `prop:char-BNT` [CPGSV16, lines 278–280 and 1135–1148], and the BNT decomposition is the subequation labelled `eq:II_ABasicTensors`:

$$A^i = X \left[\bigoplus_{j=1}^{g} (M_j \otimes A_j^i) \right] X^{-1} = \bigoplus_{j=1}^{g} \bigoplus_{q=1}^{r_j} \mu_{j,q} X_{j,q} A_j^i X_{j,q}^{-1} \qquad (\texttt{eq:II\_ABasicTensors})$$

Its matrix product vector expansion is the displayed formula labelled `decBSV`. The block-injective definition and bound are labelled `defnbi` and `propblockinj`. The proportional Fundamental Theorem is `thm1`; the equal-MPV corollary is `II_cor2`.

In the proof appendix, the normal-block overlap and phase-gauge criterion is `equalMPS`, the phase equality corollary is `eqV`, and the finite power-sum lemma is `Lem:app_simple`. These labels are the source objects that the formal BNT comparison should match. Statements about raw flattened cyclic sectors, common phase covers, or cumulative pair algebras are formal intermediates; they follow CPSV only insofar as they are proved to imply the labelled BNT comparison objects above.

1.3 Formula-by-formula translation

Reading the labelled source formulas literally gives the following translation rules for the formal theorem surface.

1. The displays `eq:II_Aiplusk1` and `eq:II_CF1` give a block diagonal tensor

$$A^i = \bigoplus_{k} \mu_k A_k^i$$

whose displayed blocks have dimensions summing to at most the original bond dimension. The formal zero-tail parameter is therefore only the dimension of the omitted all-zero summand. It is not an additional normal tensor, a BNT representative, or a separate scalar weight in the paper decomposition.

2. The phase-gauge relation `eq:II:A=XAX` [CPGSV16, lines 264–268] and the characterization `prop:char-BNT` [CPGSV16, lines 278–280 and 1135–1148] quotient the canonical-form blocks by phase and gauge. Thus the paper's BNT is a list of representatives of equivalence classes. A formal common phase cover on raw cyclic sectors is not yet the paper's BNT matching statement until it has been pushed down to these representatives.

3. The BNT expansion `eq:II_ABasicTensors` and `decBSV` separate the representative index j from the copy index q:

$$|V^{(N)}(A)\rangle = \sum_j \left(\sum_q \mu_{j,q}^N \right) |V^{(N)}(A_j)\rangle.$$

   Repeated blocks therefore appear in the paper through the power sums of the weights attached to one representative. They should not be exposed as a new family of theorem-level sector-matching statements unless the statement is exactly about recovering those multiplicities or weights.

4. The source definitions `defnbi` and `propblockinj` give exact physical-length span after blocking:

$$\operatorname{span}\{\tilde{A}^i\}_i = \bigoplus_j M_{D_j}(\mathbb{C})$$

   for the blocked physical alphabet. Cumulative algebra fullness, or membership of the identity at length zero, is not this assertion.

5. The Fundamental Theorem `thm1` compares the BNT representatives of two tensors in canonical form. The equal-case corollary `II_cor2` then uses `Lem:app_simple` on the already matched coefficient power sums. Hence the formal equal-case route should first establish the representative matching and only then recover multiplicities and weights.

These translation rules are also the naming rule for this part of the formalization. Results that merely package raw cyclic-sector data into one of these hypotheses are supporting lemmas. Theorems should be reserved for the source-level milestones above, or for final statements that explicitly include the remaining CPSV hypotheses.

1.4 Positive-length matrix product vectors

The 2016 paper defines the matrix product vector family only for positive chain lengths $N = 1, 2, \ldots$ [CPGSV16, Definition `MPV`, source lines 130–151]. Thus the source comparison is

$$\operatorname{tr}(A^{i_1}\cdots A^{i_N})=\operatorname{tr}(B^{i_1}\cdots B^{i_N})\qquad (N\geq 1),$$

or its proportional analog. A formal relation that also includes the empty word is a stronger auxiliary convention. It is useful for algebraic closure, but the source theorem does not get a positive-length conclusion from the length-zero trace.

1.5 Canonical form and period removal

The first source step removes invariant subspaces. Starting from a tensor B, the 2016 paper replaces it by block-diagonal matrices

$$A^i=\bigoplus_{k=1}^{r}\mu_k A_k^i,\qquad \sum_k D_k\leq D$$

which generate the same positive-length matrix product vectors [CPGSV16, lines 214–225]. Each block is then treated through its completely positive map. Periodic peripheral eigenvalues are removed by blocking a common multiple of their periods [CPGSV16, lines 227–231]. After that blocking, the blocks are normal in the sense used in the paper, and the tensor is in canonical form [CPGSV16, lines 233–255]. The review gives the same description: block diagonalization, period-removing blocking, primitive transfer maps, and the resulting canonical form [CPGSV21, lines 1793–1839].

This is the first important separation of tasks. Period removal is a blocking step used to make the peripheral spectrum primitive. It is not the later finite-length injectivity or block-injectivity step.

1.6 Basis of normal tensors

The next source step chooses a basis of normal tensors. In the 2016 paper, such a basis $A_1,\ldots,A_g$ is a family of normal tensors whose matrix product vectors span the matrix product vectors of A at each length and are eventually linearly independent [CPGSV16, lines 271–274]. The subsequent characterization identifies the basis tensors with representatives of the phase-gauge equivalence classes of normal blocks: every normal block in the canonical form is a phase times an invertible conjugate of one selected basis tensor, and no two selected basis tensors are related in that way [CPGSV16, lines 278–280]. The review repeats the same definition and characterization [CPGSV21, lines 1846–1859].

For a tensor in canonical form, the source then writes the matrix product vector family through the basis of normal tensors as

$$|V^{(N)}(A)\rangle=\sum_{j=1}^{g}\left(\sum_{q=1}^{r_j}\mu_{j,q}^N\right)|V^{(N)}(A_j)\rangle.$$

These are the data compared between two tensors in the equal or proportional case [CPGSV16, lines 283–302]; [CPGSV21, lines 1864–1885].

Consequently, the comparison data belong to the selected BNT representatives, not to every raw block that appears before quotienting by phase-gauge equivalence. Repeated copies and cyclic representatives contribute through the power sums of their weights.

1.7 The Fundamental Theorem and the equal case

The Fundamental Theorem itself says that, for two tensors A and B in canonical form, proportionality of all matrix product vector families matches their bases of normal tensors up to a permutation, phases, and invertible gauge matrices [CPGSV16, lines 347–352]; [CPGSV21, lines 1890–1900]. In the equal case, the source then compares the coefficients in the BNT decompositions by the power-sum identity

$$\sum_{q=1}^{r_{a,j}} \mu_{j,q}^N = \sum_{q=1}^{r_{b,j}} (\nu_{j,q} e^{i\phi_j})^N$$

for each matched basis tensor and all lengths N. The elementary power-sum lemma `Lem:app_simple` [CPGSV16, lines 1155–1163] recovers the multiplicities and weights, and the blockwise gauges assemble into the global conjugating matrix in the proof of Corollary `II_cor2` [CPGSV16, lines 1184–1192].

2 Fixed-length span after blocking

There are two different blocking operations in the source account. The period-removing blocking makes the transfer maps primitive. The Wielandt blocking supplies finite-length injectivity after normality is available. These steps should not be merged.

For one normal tensor, the quantum Wielandt theorem is cited as saying that after at most D^4 blockings the tensor becomes injective [CPGSV16, line 332]. The older canonical-form paper defines injectivity by the map

$$\Gamma_L(X) = \sum_{i_1,\ldots,i_L} \mathrm{tr}(XA_{i_1} \cdots A_{i_L}) |i_1, \ldots, i_L\rangle,$$

and observes that injectivity of Γ_L is equivalent to the length-L word products spanning the whole matrix algebra [PGVWC07, lines 888–908]. Thus the relevant conclusion is a homogeneous fixed-length span statement after a positive blocking length, not mere membership in a cumulative span. The 2021 review records a sharper MPS-specific Wielandt bound for normal tensors [CPGSV21, lines 1942–1945]; the point here is the nature of the conclusion, not which numerical bound is ultimately used.

For several basis blocks, the paper needs block-injective canonical form. In the 2016 source this is the assertion that one common physical tensor spans every block matrix simultaneously:

$$(A_1^i, \dots, A_g^i)_i \quad \text{spans} \quad \bigoplus_{j=1}^{g} M_{D_j}(\mathbb{C})$$

after the appropriate blocking [CPGSV16, lines 317–332]. The source then cites the Pérez-García–Verstraete–Wolf–Cirac bound: if L is a length at which each normal tensor is injective, then after $3(L+1)(D-1)$ blockings the canonical-form tensor is block-injective; together with $L \le D^4$, this gives the $3D^5$ bound [CPGSV16, lines 340–345]. The review states the same block-injective conclusion and bound [CPGSV21, lines 2114–2124].

The older representation paper contains the direct-sum input behind this cited step. Its condition $C1$ is exact-length injectivity of Γ_{L_0}, or equivalently length-L_0 word products spanning the full matrix algebra [PGVWC07, lines 888–908]. It then fixes canonical blocks which each satisfy $C1$, sets $L_0 = \max_j L_0^{A^j}$, orders their bond dimensions, and assumes the block states are pairwise different [PGVWC07, lines 1321–1330]. Under those hypotheses, a matrix factorization lemma and a direct-sum lemma show that the sum of the corresponding length-L MPS subspaces is direct once $L \ge 3(b-1)(L_0+1)$ [PGVWC07, lines 1333–1408]. The formal proof boundary is therefore a homogeneous fixed-length direct-sum span or its trace-dual separation form. The note `docs/paper-gaps/pgvwc07_direct_sum_input.tex` records this source input separately because it is a cited proof component, not a named theorem in the 2016 source.

This block-injective conclusion is stronger than separate injectivity of the individual blocks. It includes the cross-block separation needed to prescribe different matrices in different basis components at one common length. In dual form, one fixed word length L, after the prescribed blocking, gives the following implication. If, for every word w with $|w| = L$,

$$\sum_{j=1}^{g} \operatorname{tr}(\Delta_j A_j^w) = 0$$

then every test matrix Δ_j is zero. Equivalently, the homogeneous length-L word evaluations span the direct sum

$$\bigoplus_{j=1}^{g} M_{D_j}(\mathbb{C}).$$

This fixed-length trace-dual statement is the mathematical content behind the pair trace-separation predicates used in the non-periodic proof. It is not a separate theorem in the papers; it is the dual form of block-injective direct-sum span. In the later MPDO argument the same source explicitly writes $C_L(V) = \operatorname{span}(V^L)$ and invokes the block-injective proposition to obtain $C_L(V) = \mathcal{A}_1$ for some fixed L [CPGSV16, lines 1984–1988]. This is again an exact-length span statement.

3 The homogeneous padding point

The CPSV injectivity and block-injectivity conclusions are homogeneous. They concern words of a specified positive length after blocking. They are not consequences of the empty word lying in a cumulative word span.

This distinction matters for the pair representation used in the comparison proof. A cumulative span includes the empty word. Therefore the identity pair belongs to a cumulative span at length zero for every pair of tensors. That fact gives no positive-length padding. For example, over a one-letter alphabet with $A_0 = B_0 = 0$ in bond dimension one, the length-one pair span is generated by the zero pair, while the identity pair appears at length zero. Hence the implication from cumulative identity membership to identity membership in all later homogeneous spans is false.

Cumulative fullness alone is still too weak. In bond dimension one with one letter and pair word values $(2^n, 3^n)$, the cumulative span through length one is all of $\mathbb{C} \oplus \mathbb{C}$, since $(1, 1)$ and $(2, 3)$ are linearly independent. At each positive homogeneous length n, however, the span is only the line through $(2^n, 3^n)$, so it does not contain $(1, 1)$. Thus the source hypotheses must not be discarded down to cumulative pair-span fullness.

The source target is fixed-length block-injective direct-sum span after the prescribed blocking. A Burnside–Jacobson homogeneous-padding argument would be an alternative proof of the same kind of exact-length trace separation; it must retain the BNT, non-gauge-equivalence, and normalization data that make the fixed-length conclusion true. It is not a standalone consequence of full cumulative pair-span, and it is not a restatement of the empty-word observation.

4 Comparison after common blocking

The equal-case source proof compares BNT decompositions after the canonical form has been chosen. The after-blocking theorem reaches this situation through a longer route: discard the formal zero-tail summand just defined, choose one common physical blocking length for both tensors, reindex nested blocked words against flat words, separate period removal from later injectivity blocking, and then compare the resulting nonzero sector families.

The common-blocking question is mathematical rather than clerical. If a block has period-removal length m_k and the final common length is $p = m_k e_k$, the tensor obtained by blocking first by m_k and then by e_k has nested physical labels, whereas the tensor obtained by blocking once by p has flat word labels. Paper notation treats these identifications silently; the theorem must prove the reindexing statements.

The BNT comparison data should therefore be produced for the final nonzero sector families after all chosen blockings and reindexings. In source terms, this means the proof must construct the matched basis tensors, the phase-gauge equivalences, and the power-sum comparison for exactly those families.

The common primitive sectors should first be grouped into representative BNT classes, with repeated cyclic representatives and transported weights accounted for by the power-sum comparison. Common phase covers and proportional decomposition data match the source theorem only for those representative or grouped BNT families, not for the raw flattened cyclic-sector list. The raw list can contain several cyclic representatives of the same original block with the same transported weight, so it is not the basis of normal tensors used in the paper comparison.

5 Formal translation

The source comparison above translates into the following formal targets.

- The comparison relation should distinguish positive-length MPV equality from any auxiliary length-zero convention. If a theorem uses the empty word, the statement must say how that convention is removed before applying the paper theorem.
- The canonical-form reduction has two different blocking roles: period-removal blocking produces primitive normal blocks, while later Wielandt or block-injectivity blocking produces exact finite-length span.
- The BNT trace-separation theorem is the dual form of the fixed-length direct-sum span supplied by block-injective canonical form. The formal proof should therefore prove direct-sum span for the selected BNT representatives, or explicitly keep that fixed-length span as a hypothesis.
- Phase-cover and block-matching data must be obtained from the source's BNT comparison and the equal-case power-sum comparison. They should not be passed as unrelated assumptions on the raw flattened cyclic-sector list.
- The after-blocking comparison needs the CPSV inputs for the representative BNT families: positive dimensions, injectivity, normalization, self-overlap limits, off-overlap decay for inequivalent representatives, and equality of finite-length matrix product vector spans. The auxiliary structure used to collect these inputs is only a formal device; the mathematical obligation is to derive the listed assertions for the representatives selected by the canonical-form and BNT construction.

6 Current formal anchors

The present formalization already separates the source theorem into the same mathematical layers.

The surviving one-copy normal-canonical BNT formulation is encoded by `IsNormalCanonicalFormBNT`, together with the explicit separated-block hypotheses used by the BNT comparison lemmas.[1] This is a restricted already-grouped formulation: it suppresses the repeated-copy data $r_j, \mu_{j,q}$ that appear in the BNT expansion and Appendix BNT characterization of [CPGSV16]. The former coefficient-array structures and conditional proportional comparison theorems have been deleted. Their one-shot bounded coefficient hypotheses did not encode the residual peeling or power-sum comparison in the source paper. The former after-blocking conditional matching layer has been removed. Its hypotheses collected comparison data for the common primitive nonzero-sector families produced from the two original tensors, and its conclusion stated BNT sector matching after one positive common blocking length. The former named blocked-normal-form records and common-primitive block-matching boundary have also been removed from the formal development; this note now records only the mathematical boundary. The live anchors are the common-sector transport construction and the BNT sector-data comparison theorem.[2]

7 Named formal gaps

The current non-periodic proof route still has three CPSV obligations which should not be hidden behind similarly shaped auxiliary hypotheses.

1. Produce the common representative BNT cover data for the collapsed nonzero-sector families selected by canonical-form reduction and the BNT characterization. This is the representative-family version of the paper's phase-gauge matching theorem, not a condition on every raw block produced by an intermediate cyclic decomposition.

2. Prove the homogeneous trace-separation statement for distinct ordered BNT block pairs. In paper notation this is the trace-dual form of the block-injective span of $\oplus_j M_{D_j}(\mathbb{C})$ at one common positive length. It is not the weaker statement that the cumulative algebra generated by a pair representation is full.

3. Thread that fixed-length separation through the after-blocking comparison. The blocking used to remove periods, the blocking used to obtain injectivity, and the reindexing from nested words to flat words must be accounted for before applying the BNT-level Fundamental Theorem.

8 Source files checked

The comparison above is based on the LaTeX sources in the repository, not only on published theorem names.

[1] `TNLean/MPS/BNT/Construction.lean`.

[2] `TNLean/MPS/CanonicalForm/SectorComparison/CommonSectorTransport.lean`; `TNLean/MPS/FundamentalTheorem/SectorBNT/Fundamental.lean`.

- `Papers/1606.00608/MPDO-22-12-17-2.tex`: lines 130–151 define positive-length MPVs; lines 214–255 give canonical form and period removal; lines 271–302 define and use BNTs; lines 317–345 define injectivity, block-injective canonical form, and the $3D^5$ bound; lines 347–360 state the Fundamental Theorem and equal-case corollary; lines 1155–1163 state `Lem:app_simple`; lines 1184–1192 contain the power-sum comparison and global gauge assembly in the equal case; lines 1984–1988 use $C_L(V) = \mathrm{span}(V^L)$ as an exact-length span.

- `Papers/quant-ph_0608197/MPSarchive.tex`: lines 742–763 state the translationally invariant canonical form; lines 888–908 define Γ_L and identify its injectivity with length-L word products spanning the matrix algebra.

- `Papers/1210.6613/UncleHMPS.tex`: lines 247–278 record the standard form after blocking, including the block-diagonal span condition and its injective/block-injective special cases.

- `Papers/2011.12127/TN-Review-main.tex`: lines 1793–1900 review canonical form, BNTs, and the Fundamental Theorem; lines 1942–1945 record the normal-tensor Wielandt bound; lines 2114–2124 state the block-injective canonical-form bound.

9 Present formal boundary

The comparison with CPSV16 should be read by separating the mathematical layers of the paper proof, rather than by following implementation history. The present formalization has closed the exact-length span input for the already-injective BNT surface, and it has also formalized the Wielandt blocking input needed to make normal blocks injective after a further blocking. What remains is the transport of the final reblocked BNT cover and the coefficient comparison that CPSV16 carries out by power sums.

For an already-injective canonical BNT family, the identity-padding and pairwise trace-separation input is no longer a separate paper assumption. The formal result specializes the three-block direct-sum argument to the case in which the selected BNT blocks are one-site injective: one-site injectivity supplies the fixed length-two and length-six span inputs used by the selector construction. Thus the theorem proves the exact-length direct-sum span needed in this specialized surface, but it should not be read as the general statement with an arbitrary pre-existing injectivity length L_0. Passing from normal canonical blocks to such an injective surface is a distinct Wielandt-type blocking step.

That Wielandt step is now formalized in the canonical-form context. A left-canonical normal tensor has a positive injective blocking length bounded by D^4, and a finite normal-CF-BNT family has one common positive blocking length for all of its blocks. These theorems remove the injectivity hypothesis as a mathematical gap, but they do not by themselves identify the iterated blocked alphabet with the direct alphabet used by the after-blocking comparison. The remaining transport problem is to carry the BNT separation and zero-tail bookkeeping through this refinement of the blocking length.

The former proportional-decomposition coefficient-limit surface has been deleted rather than treated as a source theorem. CPSV16 compares the weights by the power-sum argument in the BNT proof; this step remains a genuine proof obligation. It is not a harmless coordinate convention: the block-diagonal expansion involves the powers $(\mu_j)^N$, and nonzero weights alone do not give nonzero full-sequence limits. A CPSV proof must formalize the relevant power-sum and phase comparison before feeding the result into the `BlockPermutationGaugePhaseConclusion` data and the BNT sector-data comparison.

By contrast, zero-tail equality and common-sector relabeling are mostly formal bookkeeping. CPSV16 works at positive lengths and treats zero blocks as unused bond dimension. Lean's $N = 0$ convention for $\texttt{SameMPV}_2$ forces an explicit zero-tail identity, and the common-sector construction must also track casts between grouped and flat blocked words. These conventions are not extra paper hypotheses, but the final after-blocking theorem still has to transport these identities through the same reblocking coordinates used for the nonzero BNT sectors.

The remaining main boundary is therefore the following. Starting from the normal-CF-BNT sectors produced by canonical-form reduction, one must transport blocked not-gauge-phase equivalence, zero-tail identities, and the common injective blocking length through the nested-to-flat reblocking map, and then feed the resulting families into the BNT sector-data comparison.[3] The paper suppresses this bookkeeping in notation; Lean must make it explicit.

10 Closing criterion

A formal proof obligation in this part of the theorem is discharged only when the following three assertions agree.

First, the source assertion has been identified at the level of the mathematical objects above: canonical form, basis of normal tensors, block-injective fixed-length span, phase-gauge BNT matching, or power-sum comparison. Second, the formal theorem states the same assertion for the same blocked and reindexed families, or explicitly records the extra hypothesis that remains. Third, the final comparison theorem uses that assertion rather than an unrelated assumption with a similar shape.

Conditional theorems are useful because they expose the missing mathematical inputs precisely. They should not be described as closing the source theorem until the corresponding assertion has been proved or the final theorem is deliberately restated with that hypothesis.

[3]The live target side is the sector-data theorem `MPSTensor.ft_sector_bnt_equal_sector_data`; the now-available inputs are the bounded and common injective-blocking theorems for normal-CF-BNT blocks and the already-injective direct-sum trace-separation theorems.

References

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