Source-Notation Gap: The Majumdar–Ghosh Example Tensor

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Verdict: open-gap (open).

1 The assertion

The review [CPGSV21] describes the Majumdar–Ghosh model as the one-dimensional resonating-valence-bond state, the superposition of the two nearest-neighbour singlet coverings of a periodic spin-$\frac{1}{2}$ chain. It then writes the MPS tensor in the shorthand

$$A = \big[|0\rangle\big((02| + (20|\big) + |1\rangle\big((12| + (21|\big)\big] \otimes Y, \qquad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{[CPGSV21, lines 2397–2405]}$$

where Y is the singlet tensor used earlier in the examples appendix.

The formal example file introduces instead the following bond-dimension-three matrices:

$$A^0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1/\sqrt{2} & 0 & 0 \end{pmatrix}, \qquad A^1 = \begin{pmatrix} 0 & 0 & 1 \\ -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{formal example}$$

The formalized results about this tensor are its left-canonical equation, its transfer map on the identity, non-injectivity, and non-normality.[1]

2 The point at issue

The review formula is not a literal bond-dimension-three formula. If $(ab|$ is read as $|a\rangle\langle b|$ on a three-dimensional auxiliary factor and the displayed tensor product with Y is taken literally, then the bond space is

$$\mathbb{C}^3 \otimes \mathbb{C}^2, \tag{1}$$

[1] Formal declarations: `MPSTensor.majumdarGhoshTensor`, `MPSTensor.majumdarGhosh_left_canonical`, `MPSTensor.majumdarGhosh_transferMap_one`, `MPSTensor.majumdarGhosh_not_isInjective`, and `MPSTensor.majumdarGhosh_not_isNormal` in `TNLean/MPS/Examples/MajumdarGhosh`.

so the bond dimension is 6. Writing

$$B^0 = E_{02} + E_{20}, \qquad B^1 = E_{12} + E_{21}, \tag{2}$$

the literal matrices would be $\widetilde{A}^i = B^i \otimes Y$. Their two-site periodic contractions factor:

$$\operatorname{tr}(\widetilde{A}^i \widetilde{A}^j) = \operatorname{tr}(B^i B^j)\ \operatorname{tr}(Y^2). \tag{3}$$

Since $Y^2 = -I_2$ and

$$\operatorname{tr}\big((B^0)^2\big) = 2, \qquad \operatorname{tr}\big((B^1)^2\big) = 2, \qquad \operatorname{tr}(B^0 B^1) = \operatorname{tr}(B^1 B^0) = 0, \tag{4}$$

the literal two-letter amplitudes have nonzero same-letter contractions and zero opposite-letter contractions. This is not the support pattern of a two-site singlet, whose coefficients are carried by the opposite-letter words 01 and 10 with opposite signs.

Thus the review formula must be interpreted as valence-bond shorthand rather than as a literal 6×6 MPS representation. The bond-dimension-three matrices above are the contracted representative used in the formal example: the two spin-$\frac{1}{2}$ values of an open singlet partner and the closed inter-dimer link are kept as the three bond states, and the antisymmetry of Y supplies the relative sign in A^1.

3 Current formal boundary

The repository does not yet contain a theorem identifying the matrix-product vectors of the displayed bond-dimension-three tensor with the Majumdar–Ghosh dimer superposition on even rings. Such a theorem would have to state the finite-size vector identity, with the parity, normalization, and cyclic convention fixed, for example by comparing the coefficient

$$\operatorname{tr}(A^{i_1} \cdots A^{i_N}) \tag{5}$$

with the sum of the two products of singlet coefficients on the coverings $(1,2)(3,4)\cdots$ and $(2,3)(4,5)\cdots(N,1)$.

Until that theorem is proved, the formalized object should be described as the bond-dimension-three contracted representative used for the algebraic example, not as a formally established presentation of the Majumdar–Ghosh state itself. The elementary statements already proved about it remain independent of this identification.

4 Verdict

This is an open gap in the example's physical identification. The literal reading of the review shorthand does not give the intended singlet-pair support pattern, while the contracted $D = 3$ tensor has not yet been proved in Lean to produce the Majumdar–Ghosh dimer superposition. The present formalization therefore records only the concrete tensor and its algebraic properties.

References

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.