

Kitaev Chain: the Bosonic Contraction of the Fixed-Point Matrices

2026-09-25

Verdict: scope-restriction (open).

1 What the sources print

Appendix A of [CPGSV21], under “The Kitaev chain”, describes the non-trivial fixed point of the Kitaev chain [Kit01] as a graded (fermionic) tensor network with matrices

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (1)$$

and a twist $Y = A^1$ at the boundary.¹ The state it refers to is

$$|\psi\rangle = \sum_{i_1, \dots, i_N} \text{tr}(Y A^{i_1} \dots A^{i_N}) |i_1\rangle \otimes_g |i_2\rangle \otimes_g \dots \otimes_g |i_N\rangle, \quad (2)$$

where $\otimes_g$ is the graded tensor product; the review states that this state has odd parity and that, contrary to the bosonic case, the description is injective.² The matrices and the extra y in the periodic trace are those of [BWHV17], which also notes that Y commutes with every A^i , so that the matrices read as a bosonic matrix product state are reducible.³ Section III.A.4 of the review makes the bosonic claim explicitly: the matrices commute with Y , hence the matrix product state is non-injective and can be written as a sum of two injective matrix product states, each mixing even and odd fermion numbers.⁴

2 The restriction

The graded tensor product, the sign rules of the fermionic virtual contraction, and the graded notion of injectivity are not formalized. Every formal statement

¹Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2526–2532.

²Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 474–478.

³Source traceability: `References/1610.07849/source/FermionicMPS.tex`, lines 264–289 and 372.

⁴Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 1261–1266.

about this example concerns the ordinary contraction of the matrices (1). This is enough for the coefficients of (2): they are the ordinary traces $\text{tr}(YA^{i_1} \cdots A^{i_N})$, taken relative to the ordered graded product basis. The statements proved are the following, with k the number of letters equal to 1 in the word $i_1 \cdots i_N$.⁵

1. $(A^1)^2 = -\mathbb{I}$, $A^1 = Y$, and Y commutes with A^0 and A^1 .
2. $A^{i_1} \cdots A^{i_N} = Y^k$, hence $\text{tr}(A^{i_1} \cdots A^{i_N}) = 2(-1)^{k/2}$ for even k and 0 for odd k , while $\text{tr}(YA^{i_1} \cdots A^{i_N}) = 0$ for even k and $2(-1)^{(k+1)/2}$ for odd k . Neither depends on the length N beyond k .
3. The twisted amplitudes are supported exactly on odd k , the bosonic counterpart of the odd parity in (2); the untwisted amplitudes are supported exactly on even k .
4. For every word, the $(0,0)$ and $(1,1)$ entries of the product agree and the $(0,1)$ and $(1,0)$ entries differ by a sign; these are the bosonic counterparts of the open-boundary closures discussed in [BWHV17].
5. The vector $(1, i)$ is a common eigenvector of A^0 and A^1 , the projection onto the line it spans is a nontrivial invariant orthogonal projection, and the tensor is neither injective nor normal as a bosonic matrix product state. This is the “bosonic case” that the review contrasts with the fermionic one; the fermionic injectivity claim is not formalized.
6. With a_c the tensor of bond dimension one with $a_c^0 = 1$ and $a_c^1 = c$, which is injective, and $a_{\pm} = a_{\pm i}$, $\text{tr}(YA^{i_1} \cdots A^{i_N}) = i a_+^{i_1} \cdots a_+^{i_N} - i a_-^{i_1} \cdots a_-^{i_N}$: the twisted amplitudes are a sum of two injective matrix product states. The statement that these two states are unphysical, because they mix fermion parities, is a statement about the graded setting and is not formalized.

3 Elimination plan

The restriction is lifted once graded tensor networks are formalized: a $\mathbb{Z}_2$ -graded virtual space with parity operator $\text{diag}(1, -1)$ under which A^0 is even and A^1 is odd, the graded contraction, and the graded injectivity notion of [BWHV17]. The review’s claim that the graded description is injective would then be stated against that notion.

References

- [BWHV17] Nick Bultinck, Dominic J. Williamson, Jutho Haegeman, and Frank Verstraete. Fermionic matrix product states and one-dimensional topological phases. *Physical Review B*, 95(7):075108, 2017.

⁵Results: `MPSTensor.kitaevTensor`, `MPSTensor.kitaevTensor_coeff`, `MPSTensor.kitaevTwistedCoeff_eq`, `MPSTensor.kitaevTensor_not_isNormal`, `MPSTensor.kitaevTwistedCoeff_eq_add`.

- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.
- [Kit01] A. Yu. Kitaev. Unpaired Majorana fermions in quantum wires. *Physics-Uspekhi*, 44(10S):131–136, 2001.