Example Parent Hamiltonians in Cirac–Perez-Garcia–Schuch–Verstraete 2021

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Verdict: scope-restriction (open).

1 Purpose

This note records the source scope for the example-level parent-Hamiltonian targets in issue #2315. The issue concerns the GHZ, cluster, and AKLT tensors already present in the formal development. Its mathematical aim is natural: connect these tensors to their parent Hamiltonians and to the corresponding ground-space statements. The point of this note is only to separate what is stated in the local review source from standard example formulas which need an additional citation or an explicit derivation.

2 Statements present in the review

The review gives the GHZ MPS tensor by

$$A^i_{\alpha\beta} = \delta_{i=\alpha=\beta},$$

and writes the corresponding state as

$$|\mathrm{GHZ}\rangle = \sum_{i=0}^{d-1} |i, i, \ldots, i\rangle$$

[CPGSV21, lines 2341–2345]. Earlier, in the discussion of symmetry breaking, it identifies the ferromagnetic Ising Hamiltonian

$$H = -\sum_i Z_i Z_{i+1}$$

as the paradigmatic model whose symmetric ground state is the superposition of all spins up and all spins down, and writes the same MPS tensor as $A^{\uparrow} = |0\rangle\langle 0|$ and $A^{\downarrow} = |1\rangle\langle 1|$ [CPGSV21, lines 1194–1196]. In the stability discussion it also says that the parent Hamiltonian of a GHZ state is a ferromagnetic Ising

Hamiltonian with a two-fold degenerate ground space [CPGSV21, line 2205]. The two-dimensional GHZ PEPS tensor is recorded separately in the examples appendix [CPGSV21, lines 2417–2421]. Another local source writes the two-site parent interaction explicitly as the projector complement of

$$\mathrm{span}\{|00\rangle, |11\rangle\},$$

namely, up to an additive constant,

$$\tfrac{1}{2}I - \big(|00\rangle\langle 00| + |11\rangle\langle 11|\big).$$

[1]

For the one-dimensional cluster state, the review gives the MPS tensor

$$A^0 = |0\rangle\langle +|, \qquad A^1 = |1\rangle\langle -|,$$

and explains its controlled-Z preparation from $|+\rangle^{\otimes N}$ [CPGSV21, lines 2364–2369]. The two-dimensional cluster PEPS is also displayed in the examples appendix [CPGSV21, lines 2424–2429]. In the local review source, however, the one-dimensional stabilizer Hamiltonian

$$-\sum_i Z_{i-1} X_i Z_{i+1}$$

is not stated in the one-dimensional cluster example passage. The older MPS review says instead that cluster states are unique ground states of the three-body interactions

$$\sum_i \sigma_i^z \sigma_{i+1}^x \sigma_{i+2}^z,$$

with an explicit two-matrix MPS representation [PGVWC07, local TeX lines 374–387]. Thus the issue should use the ZXZ convention for the controlled-Z cluster state unless a local-basis variant is explicitly introduced and proved equivalent. The general parent-Hamiltonian theorem for normal MPS does imply a unique ground state for a suitable parent Hamiltonian once the normal-tensor hypotheses and the range-reduction theorem have been formalized [CPGSV21, lines 2087–2090]; it is not, by itself, the explicit cluster stabilizer formula.

For the one-dimensional AKLT state, the review constructs the tensor from spin-$\frac{1}{2}$ singlets and projection onto the joint spin-1 subspace, and then gives the matrices in two bases [CPGSV21, lines 2372–2393]. In the parent-Hamiltonian section it states that the AKLT model has injectivity length $L_0 = 2$, while a two-site parent Hamiltonian is already sufficient for a unique ground state; the cited check is

$$\mathcal{G}_{1,2} \cap \mathcal{G}_{2,3} = \mathcal{G}_{1,2,3}$$

[CPGSV21, line 2095]. In the SPT discussion it also records the dangling spin-$\frac{1}{2}$ edge modes: for open boundary conditions the parent Hamiltonian has at least

[1] See `Papers/1210.6613/UncleHMPS.tex`:432–455.

the square of the virtual irrep dimension as ground state degeneracy, and for AKLT this gives a four-dimensional ground space with only one spin-0 state [CPGSV21, lines 1169–1174]. The standard description of the AKLT Hamiltonian as a sum of spin-2 projectors is not displayed in these local review passages; it should be cited to the AKLT source or derived from the spin-addition calculation before it is made a source-faithful target here. What is displayed in the local sources is the polynomial spin Hamiltonian

$$H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}\big(\vec{S}_i \cdot \vec{S}_{i+1}\big)^2,$$

together with the unnormalised matrices $\{\sigma^z, \sqrt{2}\sigma^+, -\sqrt{2}\sigma^-\}$ [PGVWC07, local TeX lines 340–349]. The same polynomial form, with an additive constant, appears in the local Schuch–Pérez-García–Cirac source.[2]

3 Current formal boundary

The formal example files already contain the tensors and their elementary structure. The GHZ tensor is defined as $A^0 = |0\rangle\langle 0|$ and $A^1 = |1\rangle\langle 1|$, and the file proves that it is an RFP, is not injective, and has the expected $\mathbb{Z}_2$ symmetry.[3] The cluster file defines the reflected one-dimensional convention $A^0 = |+\rangle\langle 0|$ and $A^1 = |-\rangle\langle 1|$, proves non-injectivity at one site, and proves two-block injectivity.[4] The AKLT file defines the spin-1 tensor and proves two-block injectivity, hence normality.[5]

The general parent-Hamiltonian API is also present. The local interaction is the orthogonal projector onto $\mathcal{G}_L(A)^\perp$, and the finite-chain parent Hamiltonian is the sum of translated local terms.[6] The theorem that the parent Hamiltonian annihilates the MPS vector is already available.[7] The normal-MPS range-reduction theorem still depends on the periodic-boundary comparison, which is the remaining formal gap for the general unique-ground-state statement.[8]

[2]See `Papers/1010.3732/paper_v3.tex`:2126–2130.

[3]See `TNLean/MPS/Examples/GHZ.lean`:54–86 and the module summary at `TNLean/MPS/Examples/GHZ.lean`:21–29.

[4]See `TNLean/MPS/Examples/Cluster.lean`:62–78 and `TNLean/MPS/Examples/Cluster.lean`:200–215.

[5]See `TNLean/MPS/Examples/AKLT.lean`:49–58 and `TNLean/MPS/Examples/AKLT.lean`:222–237.

[6]See `TNLean/MPS/ParentHamiltonian/Defs.lean`:66–75 and `TNLean/MPS/ParentHamiltonian/Defs.lean`:23–27.

[7]See `TNLean/MPS/ParentHamiltonian/Basic.lean`:118–128.

[8]Formal declaration: `wrapped_mirror_witness_agree_of_chainGroundSpace`. See `TNLean/MPS/ParentHamiltonian/UniqueGroundState.lean`:822–846 and `TNLean/MPS/ParentHamiltonian/UniqueGroundState.lean`:856–894.

4 Source-faithful order of work

A source-faithful formalization of issue #2315 should proceed in the following order.

1. For GHZ, first state the Ising parent-Hamiltonian comparison using the review terminology: the ferromagnetic Ising Hamiltonian and its two-dimensional ground space. The two-site projector description can be taken from the local uncle-Hamiltonian source.

2. For cluster, first connect the existing normality proof to the general parent-Hamiltonian construction, or prove the three-body ZXZ interaction stated in the older MPS review. Any other convention should be stated only after recording the local-basis or index-shift equivalence.

3. For AKLT, first use the source's stated two-site criterion $\mathcal{G}_{1,2} \cap \mathcal{G}_{2,3} = \mathcal{G}_{1,2,3}$ and the already formalized two-block injectivity. The polynomial AKLT Hamiltonian is locally sourced; the spin-2 projector wording should be introduced only after citing or proving its equivalence with that polynomial form. The open-boundary fourfold degeneracy should be treated separately from the periodic unique-ground-state statement.

5 Verdict

This is a scope restriction. Issue #2315 is source-faithful only when split into separate targets: the review-stated parent-Hamiltonian consequences, the explicit stabilizer or spin-projector formulas, and the open-boundary edge-mode ground spaces are distinct mathematical statements with distinct source obligations.

References

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.

[PGVWC07] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Matrix product state representations. *Quantum Information and Computation*, 7(5–6):401–430, 2007.