Conditional After-Blocking Fundamental Theorem versus the CPSV Statement

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Verdict: open-gap (open).

1 The source statement

The non-periodic matrix product vector Fundamental Theorem in [CPGSV16, CPGSV17, CPGSV21] is not stated with the auxiliary hypotheses used in the conditional Lean theorem. It is stated after the tensors have already been put in canonical form and after bases of normal tensors have been chosen. This note records that distinction for the after-blocking theorem in PR #1104, which is part of the discussion in GitHub issue #1093.

In the 2016 source, the canonical-form discussion first removes invariant subspaces. After finitely many steps the matrices are replaced by block-diagonal matrices

$$A^i = \bigoplus_{k=1}^{r} \mu_k A_k^i,$$

which generate the same family of matrix product vectors as the original tensor; zero blocks are allowed in the sense that $\sum_k D_k \leq D$ [CPGSV16, lines 214–219]. Periodic peripheral components are then removed by blocking a common multiple of their periods [CPGSV16, lines 227–231], and the paper records the resulting canonical-form existence statement: after blocking, any tensor can be represented by a tensor in canonical form generating the same family [CPGSV16, lines 249–251]. The review gives the same two-step account: first canonical form with the same family of matrix product vectors, and then the Fundamental Theorem for tensors already in canonical form [CPGSV21, lines 1752–1758]. It also describes the direct-sum canonical form and the period-removing blocking [CPGSV21, lines 1798–1820].

The comparison theorem is expressed through bases of normal tensors. In the 2016 source a basis of normal tensors is a minimal family of normal tensors such that the matrix product vectors of the original tensor are linear combinations of those basis vectors and the latter are eventually linearly independent [CPGSV16, lines 271–280]. For a tensor in canonical form, the paper then writes the matrix product vectors as sums of the basis-of-normal-tensor vectors with coefficients

$\sum_q \mu_{j,q}^N$ [CPGSV16, lines 283–302]. The review states the same basis notion and the same expansion [CPGSV21, lines 1846–1885].

Injectivity is supplied separately. The 2016 source defines injectivity and block-injective canonical form, invokes the quantum Wielandt theorem to say that normal tensors become injective after blocking, and records the block-injective bound for canonical-form tensors [CPGSV16, lines 317–344]. The review likewise states that after blocking at most $3D^5$ sites a canonical-form tensor acquires block-injective canonical form [CPGSV21, lines 2114–2124].

The CPSV statement itself is then short. For tensors A and B in canonical form, with bases of normal tensors A_{k_a} and B_{k_b}, proportionality of all matrix product vector families implies equality of the number of basis tensors and a matching of basis tensors up to permutation, phases, and invertible gauge matrices [CPGSV16, Theorem `thm1`, lines 347–352]. In the equal case the paper states the corresponding global gauge conclusion [CPGSV16, Corollary `II_cor2`, lines 354–360]. The review repeats these two statements [CPGSV21, Theorem `thm1` and Corollary `II_cor2`, lines 1891–1900]. The review also points out that a later periodic Fundamental Theorem avoids the period-removing blocking step [CPGSV21, lines 1908–1909]; that theorem is intentionally not used in the derivation considered here.

2 The conditional Lean theorem

The theorem added in PR #1104 starts from two tensors A and B with the same formal matrix product vector family. Its conclusion has the same shape as the sector-comparison part of the CPSV statement: after a positive blocking, there are sector decompositions P and Q with basis-of-normal-tensor data, the blocked tensors agree with these sector decompositions at positive lengths, the sector decompositions have the same full matrix product vector family, and the sector multiplicities and weights match up to a permutation and nonzero phases. The former blocked-normal-form records that packaged this boundary have been removed from the formal development. Mathematically, the remaining hypotheses are the comparison field for the common primitive nonzero-sector families produced by the preceding structural theorem; the blocked-word relabeling assertion for cyclic-sector data is derived in the formal theorem. The comparison field is supplied with trace-preserving, primitive, and irreducible structural data and positive bond dimensions for the produced families; it then returns BNT-cover data containing the zero-tail, injectivity, normal-form separation, and block-permutation gauge-phase comparison needed for those produced families.

These hypotheses are not written as hypotheses of Theorem `thm1` or of Corollary `II_cor2` in the cited papers. They are the formal boundary between the canonical-form construction already proved in Lean and the compressed source argument which passes from equality of total matrix product vectors to the BNT comparison theorem.

3 Paper-level inputs still to be formalized

Two parts of the named hypotheses correspond to genuine paper-level mathematics which the sources treat as part of the theorem or as standard cited input, but which is not yet derived in the present Lean proof without using the periodic Fundamental Theorem, for the specific common-sector families produced by the construction.

First, the injectivity assumptions on the produced nonzero-sector blocks correspond to the paper's use of quantum Wielandt and block injectivity. The sources do not assume injectivity in Theorem `thm1`; rather, they explain that one obtains injective or block-injective tensors after further blocking. The formal proof must still show that this refinement holds for the final common primitive sector families to which the overlap-rigidity comparison is applied.

Second, the BNT-cover hypothesis contains the paper-level comparison itself, specialized to the produced nonzero-sector families. In the source, once the tensors are in canonical form and the bases of normal tensors have been chosen, proportionality or equality of all matrix product vectors matches the basis tensors up to phases and gauges. The formal theorem in PR #1104 still takes the corresponding BNT-cover data as input for the specific common-sector families: normal canonical-form data on both sides, proof that distinct sectors are not gauge-phase equivalent, zero-tail and injectivity data, and block-matching data. A complete proof without the periodic Fundamental Theorem must derive all of this data from the equality of the original tensors' matrix product vector families, after all blockings and reindexings have been accounted for.

These two inputs are therefore not extra assumptions proposed by the papers. They are parts of the paper argument that remain to be connected to the current formal construction.

4 Coordinate and transport conditions in the formal representation

The other hypotheses are best understood as coordinate and transport data exposed by the formal representation. They are mathematical assertions, but the papers do not state them because their notation identifies the relevant objects implicitly.

The blocked-word relabeling theorem says that the nonzero part obtained by blocking once at the common length agrees with the same sector family obtained by first removing each period and then grouping the resulting blocks. In ordinary paper notation both tensors are simply regarded as tensors over words of the common blocked length. In the formal theorem, this coordinate identification is now derived from the grouped-block cast construction.

The period-removal bridge formerly exposed the analytic data used to construct cyclic sectors: a positive-definite fixed point of the adjoint transfer map, the primitive root parametrizing the peripheral eigenvalues, and irreducibility of the associated completely positive map. These quantities are now derived in the source-level Lean theorem `MPSTensor.exists_cyclic_sector_decomp_after_blocking_of_TP_of_isIrreducibleTensor`, which derives them from trace-preservation and tensor irreducibility by the quantum Perron–Frobenius theorem. Thus the blueprint period-removal theorem records the cyclic-sector decomposition itself as source-faithful. The remaining gaps in this note concern the later injectivity refinement, BNT-cover comparison, zero-tail bookkeeping, and coordinate transports for the final common-sector families.

The zero-tail condition has a similar origin. The source definition of matrix product vectors is for positive lengths, and zero blocks are invisible there. The formal relation $\mathtt{SameMPV}_2$ also includes the empty word. A zero block of bond dimension D_0 contributes nothing at positive length but contributes D_0 at length zero. Hence the formal proof records a zero-tail dimension and must either identify the two zero tails or keep the length-zero identity separate. This is a consequence of the chosen formal convention, not a hypothesis appearing in the paper theorem.

The transport of weights and zero-tail identities through the chosen common-sector representatives is likewise a coordinate-transport condition. The comparison theorem must be applied to exactly the final family of blocks after common blocking and the canonical relabeling supplied by the structural route. The paper moves between these descriptions without naming each transport map; the formal proof must show that the weights, nonzero-sector tensors, and zero-tail identities survive these changes of description.

5 Conclusion

The papers do not state the hypotheses of *the blocked-normal-form comparison boundary* as explicit theorem hypotheses. The conditional theorem in PR #1104 is therefore not yet the unconditional CPSV/CPGSV statement. It records the theorem form proved by the present derivation without the periodic Fundamental Theorem, with the remaining paper-level inputs and formal coordinate transports made visible.

This note records a proof gap in the present Lean development rather than a source error. The cited papers compress the canonical-form reduction, blocking, BNT matching, and injectivity refinements in the usual paper notation. The Lean development should keep the full blocked-normal-form comparison boundary conditional until the injectivity refinement, the BNT-cover comparison for the produced sectors, and the zero-tail and weight transports have been proved for the same final common-sector data. The analytic period-removal input itself no longer needs to be listed among the open source-faithfulness gaps.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. *Annals of Physics*, 378:100–149, 2017.

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.