

Common Blocking and Span Equality in the Non-periodic MPS Fundamental Theorem

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Verdict: clarification (resolved).

1 The compressed source argument

The proof of the non-periodic matrix product state fundamental theorem passes from arbitrary tensors with the same matrix product vectors to basis-of-normal-tensors comparison data. The source argument is the MPDO canonical-form paper [CPGSV16, CGSV17], together with the later review [CPGSV21].¹ The cited papers are not being questioned. The point is that a formal proof has to separate several canonical-form steps which are compressed in the source prose.

For a tensor $A = (A^i)_{i=1}^d$, the source paper defines the matrix product vectors by

$$|V^{(N)}(A)\rangle = \sum_{i_1, \dots, i_N=1}^d \text{tr}(A^{i_1} \dots A^{i_N}) |i_1\rangle \otimes \dots \otimes |i_N\rangle,$$

for $N = 1, 2, 3, \dots$ [CPGSV16, definition of the matrix product vector family, lines 130–138]. After removing invariant subspaces, the same paper writes the tensor in block form

$$A^i = \bigoplus_{k=1}^r \mu_k A_k^i,$$

allowing zero blocks and scaling the nonzero blocks so that their transfer maps have spectral radius one. Each nonzero irreducible block may still have peripheral period. Blocking by a common multiple of the periods removes that periodicity [CPGSV16, lines 214–231]. The review follows the same canonical-form argument [CPGSV21, lines 1798–1820].

The later comparison is made through bases of normal tensors. A basis of normal tensors is a minimal family $(A_j)_{j=1}^g$ of normal tensors such that every matrix product vector of A is a linear combination of the matrix product

¹For traceability, this note corresponds to issue #1047.

vectors of the A_j , and such that those vectors are eventually linearly independent. Equivalently, every normal block in the canonical form is a phase times a gauge transform of exactly one basis tensor, with the family chosen minimally [CPGSV16, Proposition `prop:char-BNT`, lines 271–280]. The review states the same characterization [CPGSV21, Proposition `prop:char-BNT`, lines 1846–1859]. For a tensor in canonical form one then has an expansion

$$|V^{(N)}(A)\rangle = \sum_{j=1}^g \left(\sum_{q=1}^{r_j} \mu_{j,q}^N \right) |V^{(N)}(A_j)\rangle.$$

This is the basis-of-normal-tensors expansion used in the equal-case proof [CPGSV16, lines 283–302]. The review uses the same expansion [CPGSV21, lines 1864–1885].

Injectivity enters at a different point. The source first defines injective and block-injective canonical form, and then invokes the quantum Wielandt theorem to say that, after a bounded further blocking, every normal tensor becomes injective [CPGSV16, lines 317–332]. The review records the corresponding $3D^5$ bound for block-injective canonical form [CPGSV21, lines 2120–2124]. This step is not the same as period removal: period removal gives primitive normal blocks, while the overlap-rigidity arguments used later require one-site injectivity, or an explicit replacement after further blocking.

Finally, the source comparison for the equal case is compact. Equality or proportionality of the matrix product vector families first matches the two bases of normal tensors by phase and gauge. After relabeling, the coefficient multiplicities are compared through the power sums

$$\sum_{q=1}^{r_{a,j}} \mu_{j,q}^N = \sum_{q=1}^{r_{b,j}} (\nu_{j,q} e^{i\phi_j})^N, \quad \text{for all } j \text{ and } N.$$

The elementary power-sum lemma then recovers the multiplicities and the weights, and the blockwise gauges are assembled into the global conjugating matrix [CPGSV16, lines 1181–1192]. The review states the resulting proportional and equal MPV fundamental theorems in the same form [CPGSV21, lines 1891–1900].

2 The formal argument

The canonical-form reduction uses the positive-length relation `SameMPVPos`, matching the family in the source definition. All-zero blocks contribute nothing at positive length and are discarded. The total bond dimension of the retained family is bounded directly by the structural direct-sum identity, without an empty-word comparison.²

²See `exists_irreducible_blockDecomp_nonzeroBlocks` and `exists_tp_gauge_from_arbitrary`.

The common blocking step is also more explicit. For a single family of blocks, the source phrase “block by the least common multiple of the periods” hides no mathematical difficulty. In the two-tensor comparison, however, the formal proof must choose one positive physical blocking length for both sides. For each original block with period-removal length m_k , the common length is written as $p = m_k e_k$. The tensor obtained by first blocking by m_k and then by e_k has physical labels given by nested words, while the tensor obtained by blocking once by p has physical labels in the alphabet of words of length p . The formal statement therefore keeps the relabeling between these two alphabets as part of the statement.³ This is why the canonical-form theorem supplies relabeled common-sector families and a conditional label-compatibility statement, rather than immediately replacing every canonical nonzero part after blocking by its flattened sector tensor.

The formal comparison theorems downstream now use the coefficient-comparison route for BNT sectors. The live data are the matched-sector phase relation, the coefficient identities extracted from eventual linear independence, the copy-weight matching obtained from the power sums, and the block-diagonal global gauge. Any additional Wielandt blocking needed for one-site injectivity is a separate refinement for arguments that require injective blocks; it is not part of the current coefficient-comparison theorem.⁴

The paper’s compact BNT matching and power-sum comparison is no longer represented by the removed comparison lemmas from the former span-based route. The current formal comparison follows the source proof more directly: matched BNT sectors give the phase relation

$$V^{(N)}(Q_k)_\sigma = \zeta_k^N V^{(N)}(P_{\beta(k)})_\sigma,$$

eventual linear independence gives the coefficient identities

$$c_N^{(\beta(k))}(P) = \zeta_k^N c_N^{(k)}(Q),$$

Newton–Girard power-sum comparison recovers the copy weights, and the matched block gauges assemble into the direct-sum global gauge.⁵ The

³The common-sector data and the relabeling results are recorded in `blueprint/src/chapter/ch09_canonical.tex:1645--1824` and `blueprint/src/chapter/ch09_canonical.tex:3230--3284`. The main formal statement is `MPSTensor.fundamentalTheorem.after.blocking.commonLength.commonSector`; see `TNLean/MPS/CanonicalForm/SectorComparison/CommonSectorData.lean` and `TNLean/MPS/CanonicalForm/SectorComparison/CommonSectorTransport.lean`.

⁴The final comparison route is recorded in `blueprint/src/chapter/ch11_fundamental_theorem_proof.tex:31--460`.

⁵See `blueprint/src/chapter/ch11_fundamental_theorem_proof.tex:31--460`, especially the sector coefficient identity, copy-weight matching, and assembled global-gauge entries. The corresponding formal objects live in `TNLean/MPS/FundamentalTheorem/SectorBNT/CoeffIdentity.lean`, `TNLean/MPS/FundamentalTheorem/SectorBNT/CopyWeightMatching.lean`, `TNLean/MPS/FundamentalTheorem/SectorBNT/Fundamental.lean`, and `TNLean/MPS/FundamentalTheorem/SectorBNT/FundamentalCoord.lean`. The removed comparison lemmas from the former span-based route are no longer used in the final comparison chapter; the remaining proof follows this coefficient-comparison route.

coefficient-identity and copy-weight lemmas state precisely what comparison data are needed; the proportional case still requires the coefficient comparison for the phases supplied by the sector matching theorem.

3 Why these distinctions belong together

These distinctions are coupled in the single conversion from an arbitrary equality of MPV families to the hypotheses of the BNT sector-comparison theorem. The common blocking construction determines the physical alphabet and the relabeled sector families to which the coefficient comparison applies. Any later injectivity refinement is reserved for separate arguments, such as overlap-rigidity statements that require one-site injective blocks; it is not a hypothesis of the SectorBNT coefficient route. The matched-sector coefficient identities and copy-weight data are constructed for the relabeled nonzero-sector families, not for an earlier family before the common-alphabet relabeling.

For this reason common blocking and coefficient comparison should be kept in one account. The formal proof needs the whole chain: discard all-zero blocks at positive lengths, choose one common physical blocking length, account for the word-relabeling, keep later overlap-rigidity requirements separate, and turn BNT matching data into the sector coefficient identities and copy-weight equations used by the final gauge comparison. The blueprint states this chain as the connection between the canonical-form chapter and the final comparison chapter.⁶

This note should also not be read as a claim that the same-side equal-norm scalar obstruction discussed in the broader fundamental-theorem work is a flaw in the CPSV or CPGSV arguments. That obstruction corrected an over-strong formal statement which tried to identify an equal-norm class with a single basis tensor. The BNT expansion in the papers already allows multiple basis tensors and multiplicities. The formal argument therefore moved from naive norm-class grouping to phase-class sector decompositions and common-cover data.

4 Conclusion

The conclusion is that the cited papers compress several standard canonical-form operations; no counterexample is claimed. Zero or nilpotent contributions are invisible to positive-length MPVs, periods are removed by blocking, injectivity is obtained by a later Wielandt-type blocking, BNT bases are matched by phase and gauge, and coefficient multiplicities are recovered from power sums. The formal development exposes these operations as separate statements because the formal MPV relation is transported through explicit positive blocking, the

⁶See `blueprint/src/chapter/ch11_fundamental_theorem_proof.tex:539--629` for the current equal and proportional BNT canonical-form statements. The proportional global gauge remains conditional on the coefficient identity recorded at `blueprint/src/chapter/ch11_fundamental_theorem_proof.tex:361--388`; the equal case uses `blueprint/src/chapter/ch11_fundamental_theorem_proof.tex:503--537`.

two sides must be placed over one explicitly relabeled alphabet after blocking, and the downstream comparison theorem constructs matched-sector coefficient identities and copy-weight equations after this relabeling.

Thus the comparison gives neither a counterexample nor a theorem-level source error. It explains why the formal proof cannot compress the argument into a single invocation of the BNT uniqueness theorem. The relevant comparison points are removal of all-zero blocks, common primitive block construction, common blocking, matched-sector coefficient identities, copy-weight matching, and iterated-blocking relabeling.⁷

References

- [CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: *Annals of Physics* 378 (2017), 100–149, 2016.
- [CPGSV17] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. *Annals of Physics*, 378:100–149, 2017.
- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.

⁷For traceability, these are recorded in issue #652, issue #942, issue #969, issue #970, issue #971, issue #944, and issue #990.