

The Sign of σ_y in the Pauli Form of the AKLT Tensor

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Verdict: local-correction (resolved).

1 The source statement

The examples appendix of Ref. [CPGSV21], subsection “The AKLT state”, prints the AKLT tensor with the physical states labelled by $S_z = 0, \pm 1$ as

$$A^{+1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Y, \quad A^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Y, \quad A^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Y, \quad (1)$$

with the singlet matrix

$$Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2)$$

It then states that in the basis

$$|+\rangle = \frac{i}{\sqrt{2}} (|-1\rangle + |+1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}} (|-1\rangle - |+1\rangle), \quad |0\rangle, \quad (3)$$

the tensor becomes $A^- = \sigma_x/\sqrt{2}$, $A^+ = \sigma_y/\sqrt{2}$, and $A^0 = \sigma_z/\sqrt{2}$.¹

2 The computation

Write $|\psi\rangle = \sum_s \text{tr}(A^{s_1} \cdots A^{s_N}) |s_1 \cdots s_N\rangle$. The basis (3) is orthonormal, so the component of $|\psi\rangle$ along $|a_1 \cdots a_N\rangle$ is obtained from the tensor

$$B^a = \sum_s \langle a | s \rangle A^s. \quad (4)$$

The matrices in (1) are $A^{+1} = -|0\rangle\langle 1|$, $A^0 = \sigma_z/\sqrt{2}$, and $A^{-1} = |1\rangle\langle 0|$. Since $\langle + | \pm 1 \rangle = -i/\sqrt{2}$,

$$B^+ = -\frac{i}{\sqrt{2}} (A^{-1} + A^{+1}) = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\frac{1}{\sqrt{2}} \sigma_y, \quad (5)$$

¹Source traceability: `Papers/2011.12127/TN-Review-main.tex`, lines 2372–2393.

while $B^- = (A^{-1} - A^{+1})/\sqrt{2} = \sigma_x/\sqrt{2}$ and $B^0 = \sigma_z/\sqrt{2}$ agree with the source. The printed σ_y is the value of $\sum_s \langle s | a \rangle A^s$, which uses the complex conjugate of the correct coefficient.

The discrepancy is a virtual gauge together with a scalar. Conjugation by σ_y reverses the signs of σ_x and σ_z and fixes σ_y , so

$$(\sigma_x, -\sigma_y, \sigma_z) = \sigma_y \left(-(\sigma_x, \sigma_y, \sigma_z) \right) \sigma_y^{-1}. \quad (6)$$

The reversal of orientation caused by the sign of σ_y is undone by the overall factor -1 . Hence on N sites the state described by $(\sigma_x, -\sigma_y, \sigma_z)/\sqrt{2}$ is $(-1)^N$ times the state described by the printed form $(\sigma_x, \sigma_y, \sigma_z)/\sqrt{2}$: the two states coincide up to a global sign. For $N = 3$ every nonzero coefficient changes sign; for instance the coefficients of $|- , + , 0 \rangle$ are $\text{tr}(\sigma_x \sigma_y \sigma_z)/2^{3/2} = i/\sqrt{2}$ and its negative.

3 The correction

In the printed basis (3) the component of the tensor along $|+\rangle$ is $-\sigma_y/\sqrt{2}$. By (6) this tensor equals the printed Pauli form up to the virtual gauge σ_y and the scalar -1 , and its state equals the state of the printed Pauli form up to the global sign $(-1)^N$. The printed Pauli form holds exactly in the orthonormal basis

$$|+\rangle = -\frac{i}{\sqrt{2}} (|-1\rangle + |+1\rangle), \quad (7)$$

with $|-\rangle$ and $|0\rangle$ unchanged. Every symmetry, normality, and parent-Hamiltonian property of the Pauli form is unaffected. The component identities in both bases, the gauge-and-sign relation (6), the resulting factor $(-1)^N$ on the state coefficients, and the orthonormality of both bases are formally verified in `TNLean/MPST/Examples/AKLTReview`.²

4 Verdict

The printed Pauli form of the AKLT tensor in Ref. [CPGSV21] describes the AKLT state up to a global sign: in the printed basis the tensor is $(\sigma_x, -\sigma_y, \sigma_z)/\sqrt{2}$, which is the printed form up to the virtual gauge σ_y and the scalar -1 . As an identity of tensors, the printed form holds with the phase of $|+\rangle$ conjugated. The tensor itself, its derivation from singlet bonds, and the statements about A^- and A^0 hold as printed.

²In particular, `MPSTensor.rotatePhysical_akltRMPBasis`, `MPSTensor.akltTensorRMPPauli_neg_gaugeEquiv`, `MPSTensor.mpv_rotatePhysical_akltRMPBasis`, and `MPSTensor.rotatePhysical_akltRMPBasisFixed`.

References

- [CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.