Scope Note: The AKLT On-Site SO(3) Symmetry Is Formalized; Its Cohomological Classification Is Not

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Verdict: scope-restriction (open).

1 The assertion

The review [CPGSV21] records that the AKLT matrix product state lies in a non-trivial symmetry-protected topological phase, and that this phase "can be protected by multiple distinct physical symmetries: on-site SO(3) (with virtual symmetry given by a spin-$\frac{1}{2}$ representation), on-site $\mathbb{Z}_2 \times \mathbb{Z}_2$ (the smallest subgroup of SO(3) still exhibiting a non-trivial 2-cocycle), time-reversal, or reflection symmetry."[1] In each case the physical symmetry acts on the spin-1 site, while the entanglement (virtual) degrees of freedom carry a projective representation whose cohomology class is the order-two generator of H^2 of the symmetry group with U(1) coefficients.

The strongest such statement is on-site SO(3): the spin-1 representation acts on the physical index, and the virtual index transforms in the projective spin-$\frac{1}{2}$ representation, whose class generates $H^2(\mathrm{SO}(3), \mathrm{U}(1)) \cong \mathbb{Z}_2$.

2 The point at issue

Two different statements must be separated: the on-site SO(3) *symmetry*, and the cohomological *classification* of the phase it protects. They have very different formalization costs.

[1] [CPGSV21], around line 1159 of the source.

The symmetry. This statement is now formalized. For every rotation $R \in \mathrm{SO}(3)$ the Cartesian AKLT tensor $A^i = \sigma^i$ twisted by R is gauge equivalent to the original, the gauge being an SU(2) preimage of R.[2] The one substantive input is the surjectivity of the adjoint double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$, that every rotation is realized by conjugating the Pauli vector $(\sigma_x, \sigma_y, \sigma_z)$ by an SU(2) element. It is proved by an explicit ZXZ Euler-angle construction: each coordinate rotation is the image of an explicit SU(2) generator, and an arbitrary rotation is factored into coordinate rotations.[3] The physical spin-1 representation is just the defining action of SO(3) on the three Cartesian components, and the adjoint map $U \mapsto R(U)$ is a monoid homomorphism, so its image is automatically a subgroup; hitting the Euler generators therefore yields surjectivity with no appeal to Lie-group topology, to the Clifford spin group, or to a quaternion bridge (none of which the library provides).

The classification. The claim that the resulting projective phase is the order-two generator of $H^2(\mathrm{SO}(3), \mathrm{U}(1)) \cong \mathbb{Z}_2$ is a different matter entirely. It requires the continuous (Borel, or Lie-group) cohomology of SO(3), equivalently its Schur multiplier. The group cohomology in the library is purely algebraic and treats SO(3) as a discrete group, so it computes the wrong object; as of this writing there is no Schur multiplier and no projective-representation interface, and the nascent continuous-cohomology framework has no computations and is not yet reconciled with the discrete theory. This piece is a substantial development on its own.

3 The formalized statement

The formalization establishes the full on-site SO(3) symmetry of the Cartesian AKLT tensor $A^i = \sigma^i$: the physical site carries the spin-1 (defining) representation, and for each rotation the bond index is conjugated by a spin-$\frac{1}{2}$ SU(2) preimage. The finite $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup, which the review singles out as the smallest subgroup of SO(3) carrying a non-trivial 2-cocycle, is formalized separately on the $|m\rangle$-basis tensor with the explicit anticommuting gauges $i\sigma_y$ and σ_z.[4]

In both cases the bond representation is projective: rotations that commute have SU(2) preimages that need not commute, and that sign obstruction is the content of the symmetry-protected phase. For $\mathbb{Z}_2 \times \mathbb{Z}_2$ the ob-

[2] `aklt_isOnSiteSymmetric_SO3` in `TNLean/MPS/Examples/AKLTRotation`.
[3] `spinHalfCover_surjective_onto_SO3`.
[4] `aklt_isOnSiteSymmetric_Z2Z2` in `TNLean/MPS/Examples/AKLT`.

struction is the explicit anticommutator $i\sigma_y \, \sigma_z = -\sigma_z \, i\sigma_y$; for the full SO(3) it is the non-triviality of the double cover.

4 Conclusion

The on-site SO(3) symmetry asserted by [CPGSV21], together with the surjectivity of the spin-$\frac{1}{2}$ cover that it rests on, is now formalized. What remains unformalized is the *cohomological classification* of the resulting projective phase as the order-two generator of $H^2(\mathrm{SO}(3), \mathrm{U}(1)) \cong \mathbb{Z}_2$: this needs the continuous group cohomology of SO(3) (equivalently its Schur multiplier), which the library does not provide. The blueprint therefore tags the symmetry and the cover-surjectivity against their formal declarations, and records the cohomological classification as an untagged remark.

References

[CPGSV21] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Reviews of Modern Physics*, 93(4):045003, 2021.