

Physical-Port Simulation of Bounded Register Channels

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Verdict: scope-restriction (open).

1 The source resource convention

Pirolì, Styliaris and Cirac [PSC21], Supplement pp. 7–8, distinguish counted nearest-neighbor gates on fixed-dimensional physical qudits from free onsite operations involving those qudits and their local ancillas. In the proof of Proposition 2, the entanglement increase per intersite layer depends on the physical dimension, independently of the number of onsite ancillas. This prevents replacing a physical gate by an arbitrary operation on two enlarged sites at the same unit cost without a simulation argument.

The existing enlarged-site local-channel model allows arbitrary intermediate local dimensions. Its composition and trace-norm estimates remain statements about that model. The construction below supplies a bounded-register communication theorem; it is not an inclusion theorem for unrestricted intermediate dimensions.

The depth used here counts only physical-port intersite layers. In the main text, p. 1, the source defines depth by nearest-neighbor pair-gate layers and treats local unitaries on each spin and its ancillas as free. The displayed constants therefore do not count onsite operations as extra gate layers. Every finite protocol in this reduced-channel model now has an explicit unitary realization with fresh local product-zero environments, unchanged physical ports and intersite depth, and one final environment trace.

2 The proved bounded matching theorem

Fix a physical dimension $d \geq 2$, an upper bound B on the local input dimension, and $m \leq B$. The density witness below forces $m \geq 1$, hence $B \geq 1$. Set

$$k = \lceil \log_d B \rceil. \tag{1}$$

The local basis code into k qudits is chosen independently of the chain length. Its matrix E is isometric. A completely positive trace-preserving decoder is

obtained by completing $Y \mapsto E^\dagger Y E$ outside the code:

$$D(Y) = E^\dagger Y E + \text{tr}[(I - E E^\dagger)Y(I - E E^\dagger)]\rho, \quad (2)$$

where ρ is a fixed density matrix. Then $D(E X E^\dagger) = X$ for every operator X , and the identity persists after tensoring with the identity on any finite reference. The two-site encoding is the tensor product of the separate onsite encodings.¹

On a ring of $N \geq 2$ sites, give each site one physical port, a k -qudit data register and a k -qudit scratch register. For an arbitrary matching of CPTP pair channels, extend each channel to its full two-register space by encoding its output after applying it to a completed decoding of the input. These extensions are CPTP on the entire register algebra and agree exactly with the original channels on encoded inputs. There are normalized finite Kraus representations whose actual placed matching channel has physical depth at most

$$2k = 2\lceil \log_d B \rceil, \quad (3)$$

independently of N and the number of selected bonds. Products of L such certified layers have depth at most $2kL$.² For $B = 1$, the data register has zero qudits and the bound is zero. A one-site ring's self-loop SWAP is separately proved to be the identity; the two-distinct-register statement assumes $N \geq 2$.

3 Why the communication cost is uniform

To exchange one memory digit across a selected bond, swap that digit with the physical port locally at both endpoints, perform one physical-port SWAP, and undo the two local swaps. This exchanges the selected memory digits and restores the ports exactly, including when they carry arbitrary correlations. Every bond of a matching runs in parallel. Repeating over the k digits moves each left data register into the right scratch register, while leaving the right data register in place.

All pair channels can now be executed onsite. Reverse the one common routing afterward. The total cost is $k + k$, not $2k$ times the number of bonds. Conjugation by the routing preserves products of channel maps, so the resulting map is exactly the original placed family on the complete operator space. Outputs return to their original data registers and all other wires are restored.³ This construction follows the same gather, execute locally, and return principle used in the source's proof of Theorem 1 (Supplement pp. 8–9), but the displayed channel-resource bound is a derived result.

¹Formal declarations: `QuantumCircuit.registerDecoding_comp_encoding`, `QuantumCircuit.registerDecoding_encoding_reference`, and `QuantumCircuit.registerEncoding_pair` in `TNLean/Circuit/Channel/RegisterEncoding`.

²Formal declarations: `QuantumCircuit.exists_bounded_matching_simulation`, `QuantumCircuit.PortRegisters.matchingChannel_isPhysicalPortProtocol`, and `QuantumCircuit.IsPhysicalPortProtocol.list_prod_uniform` in `TNLean/Circuit/Channel/PortSimulation`.

³Formal declarations: `QuantumCircuit.IsPhysicalPortUnitary.register_matching` in `TNLean/Circuit/Channel/PortRouting` and `QuantumCircuit.IsPhysicalPortProtocol.of_routed_family` in `TNLean/Circuit/Channel/PortChannelRouting`.

The completed decoder for a joint code need not factor into separated onsite operations outside that code. No such assertion is used: the joint extension is applied only after the registers have been colocated. Separate input/output encoding and decoding use products of local maps. Decoding after encoding is the identity on every chain operator, with any finite reference.⁴

4 The encoded bounded-protocol compiler

The dimension bound now records every positive intermediate local space of an actual native protocol, not just its endpoint dimensions. For fixed $d \geq 2$, $N \geq 2$, and one common bound B , set $k = \lceil \log_d B \rceil$. A native protocol Ψ of depth T compiles into an explicit word on the alphabet d^k . Its onsite operations cost zero and its T native matching layers cost at most $2kT$ in total.⁵

The resulting physical channel Γ obeys the exact identity

$$\Gamma \mathcal{E}_{\text{in}} = \mathcal{E}_{\text{out}}(\Psi \otimes \text{id}_R), \quad (4)$$

where R is the entire complementary port/scratch system. The equation holds on all joint input-reference operators. The maps $\mathcal{E}$ are product onsite codes transported through explicit sitewise digit and selected/complementary coordinate equivalences. Encoders and completed decoders are CPTP, and decoding after encoding is exactly the identity. The coordinate changes are not assumed free intersite gates.

This is an encoded logical-input statement: the data occupy local data registers. The next construction supplies the original-port input/output interface when the native input and output dimensions both equal d .

5 Original-port input, output, and designated memory reset

Let J_{port} be the explicit product code that stores the input qudit on each original physical port and initializes its $2k$ data/scratch digits to zero. A second product code stores the register encoding on the data digits, with zero port and scratch digits. Completed channels on each whole spatial-site block convert these codes in both directions at zero intersite depth. They are CPTP on the entire physical operator space, not merely on the code.

Compose this preprocessing and postprocessing with the encoded compiler. For a bounded native endomorphism Ψ whose input/output dimension is d , the resulting physical-port protocol satisfies

$$\Gamma_{\text{port}} J_{\text{port}} = J_{\text{port}} \Psi, \quad \text{depth}_{\text{intersite}}(\Gamma_{\text{port}}) \leq 2kT. \quad (5)$$

⁴Formal declarations: `QuantumCircuit.OnsiteChannel.decodeRegisters_encodeRegisters_reference` in `TNLean/Circuit/Channel/OnsiteRegisterEncoding`.

⁵Formal constructive compiler: `QuantumCircuit.IsDimensionBoundedLocalChannelProtocol.exists_physicalPortSimulation` in `TNLean/Circuit/Channel/PhysicalProtocolCompilation`.

Thus the output returns to the same physical ports, and all designated data/scratch memory is again in its product zero state and uncorrelated with the output. The identity holds on all logical operators and persists with an arbitrary external reference.⁶

The explicit initializer is identified with the earlier physical encoding by splitting data from its complementary wires and preparing the latter in their zero configuration. This identity is checked on matrix units, so it includes off-diagonal coherences. The reset assertion concerns the designated data/scratch wires. The additional dilation environments in the unitary realization below are discarded; they are not asserted to reset.

6 One unitary circuit and one final environment trace

Let P assign the W existing d -level wires to $N \geq 1$ spatial sites and designate one communication port at each site. For every finite depth- T physical-port channel protocol Φ , with $d \geq 1$, there are finitely many appended environment wires, an explicit assignment of each new wire to a spatial site, and an actual unitary circuit U on the enlarged layout such that

$$\mathcal{D}_E \circ \text{Ad}_U \circ \mathcal{I}_E = \Phi, \quad \text{depth}_{\text{intersite}}(U) \leq T. \quad (6)$$

Here $\mathcal{I}_E(X) = X \otimes |0_E\rangle\langle 0_E|$ in the explicit original/environment coordinates, $\mathcal{D}_E$ discards exactly the appended wires, and $\text{Ad}_U(Y) = UYU^\dagger$. Every original wire retains its spatial owner, and the designated physical ports are unchanged. All environments are initialized at the start and remain in the joint system until the one final discard. The identity holds on all original-wire operators, including their arbitrary external references.⁷

For a channel on m qudits within one spatial site, $2m$ fresh qudits at that same site suffice. The retained output is the original local register; all other existing wires are untouched, including their correlations with that register. This uses the d^{2m} Choi-rank bound, followed by a unitary extension of the fixed pure-input isometry. For composition, the two circuits use disjoint fresh environment blocks. The second circuit acts as the identity on the earlier environment, so its trace can be postponed even when that environment is correlated with the system. Only the original physical ports participate in counted intersite layers.⁸

⁶Formal theorem: `QuantumCircuit.IsDimensionBoundedLocalChannelProtocol.exists_physicalPortIO` in `TNLean/Circuit/Channel/PhysicalPortIO`.

⁷Formal declarations: `QuantumCircuit.PhysicalPortEmbedding.unitary` in `TNLean/Circuit/Channel/PhysicalPortEmbedding`, and `QuantumCircuit.IsPhysicalPortProtocol.exists_appended_unitary_dilation`, `QuantumCircuit.IsPhysicalPortProtocol.exists_appended_unitary_dilation_reference` in `TNLean/Circuit/Channel/PhysicalPortDilation`.

⁸Formal declarations: `QuantumCircuit.exists_selected_onsite_dilation` and `QuantumCircuit.exists_onsite_appended_dilation` in `TNLean/Circuit/Channel/PhysicalPortLocalDilation`; `QuantumCircuit.appendEnvironment_comp_channel` in `TNLean/Circuit/Channel/PhysicalPortEnvironmentComposition`. The finite-dimensional dilation and deferred-partial-trace identities are proved in the companion QICLean library.

Applying (6) to the bounded original-port protocol gives a unitary U satisfying

$$\begin{aligned}\mathcal{D}_E \circ \text{Ad}_U \circ \mathcal{I}_E \circ J_{\text{port}} &= J_{\text{port}} \circ \Psi, \\ \text{depth}_{\text{intersite}}(U) &\leq 2\lceil \log_d B \rceil T.\end{aligned}\tag{7}$$

Thus both the designated memory and the additional environments start in product-zero states. After the final environment trace, the output is on the original ports and the designated data/scratch memory is reset to zero. The discarded environment may contain information about the input; no reset of that environment is claimed.⁹ The total number of environment wires is finite for each protocol; this existence theorem does not impose a uniform environment-size bound on an entire family.

7 Remaining comparison and source conditions

The finite local-unitary dilation, product-zero environment initialization, and one final disposal are proved by (6). A full source protocol witness must still satisfy the source’s final system–ancilla condition when claiming a pure-state preparation, together with its measurement and classical-control rules. No clean unitary identity that returns every dilation environment to zero is asserted. For the asymptotic $QCcc$ relation, the source fixes the number of composed $QCcc$ protocols independently of N . This is not a bound on individual local measurement/correction steps: one protocol can already contain sequential steps at many sites. A source-block decomposition must also satisfy the within-block control order and one-measurement-per-site restriction. None of these conditions follows from a quantum-depth bound.

Every finite positive-input native protocol admits some finite dimension budget. For a family this gives only a potentially size-dependent $B(N)$, which does not supply the uniform constant above. The encoded compiler therefore does not identify the unrestricted enlarged-site phase relation with the source’s relation. No MPS classification is inferred.¹⁰

References

- [PSC21] Lorenzo Piroli, Georgios Styliaris, and J. Ignacio Cirac. Quantum circuits assisted by local operations and classical communication: Transformations and phases of matter. *Phys. Rev. Lett.*, 127:220503, 2021.

⁹Formal theorem: `QuantumCircuit.IsDimensionBoundedLocalChannelProtocol.exists_physicalPortUnitaryIO` in `TNLean/Circuit/Channel/PhysicalPortUnitaryCompilation`.

¹⁰The resource comparison is tracked by GitHub issue #8640, under GitHub issue #8622; approximate and adaptive conversions are tracked by GitHub issue #8380 and GitHub issue #8382.