

The Polynomial Count of the Small-Patch Rewrite

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Verdict: scope-restriction (resolved).

1 The source lemma

Lemma 6.3 of the polynomial PEPS approximation manuscript of September 24, 2026 [Ope26] (`05-frames.tex`, lines 188–219) considers old and new one-sheet encoded frames that agree outside boundedly many affected holes, and m additional square patches satisfying four conditions: (i) the patch outer squares and the affected holes avoid the outer footprints of the untouched holes; (ii) the patch inner squares cover every site whose raw owner changes and the whole outer footprint of every affected hole; (iii) in each patch outer square, the sites outside the old inner holes have at most two old owners, and the sites outside the new inner holes have at most two new owners; (iv) all involved raw owners and affected tag owners belong to a specified bounded list of parties. The lemma asserts:

1. the canonical map $M = K_{\text{new}} P_m \cdots P_1 K_{\text{old}}^\dagger$ is a contraction with $\|M\Omega_{\mathcal{F}_{\text{old}}} - \Omega_{\mathcal{F}_{\text{new}}}\| \leq (m + r_{\text{old}})\varepsilon$;
2. for every fixed $a > 0$, M has a contractive approximation M_a with $\|M_a - M\| \leq L^{-a}$ and a polynomial expansion into monomials allowed by Theorem 5.2, using only the specified parties.

2 What the formal statement covers

The first assertion is formalized with exactly the source hypotheses, for frames whose tags are listed in any order; the reference error uses only condition (i).¹ For the second assertion the following steps of the proof (`05-frames.tex`, lines 254–341) are formalized: the exact expansion of M into complete branches

¹`TNLean.PEPS.EncodedFrame.SmallPatchRewrite.norm_rewrite_le_one_and_refVec_sub_le`, `TNLean.PEPS.EncodedFrame.SmallPatchRewrite.norm_rewrite_refVec_sub_le`, and, for arbitrary tag orderings, `TNLean.PEPS.EncodedFrame.SmallPatchRewrite.exists_norm_le_one_and_refVec_sub_le_of_perm`.

and the count of branches; the site-level wire classification of every branch, for every choice of cylinder terms, including unequal radii, holes and patches with empty outer sample (whose identity term selects the empty square), and overlapping patches and holes; the summation of branchwise errors; the choice of the truncation rank k ; and the final rescaling by $1 + \frac{1}{2}L^{-a}$.

The residual network of a branch is also formalized. For a branch (a, s, b) , the raw operator $B_a C_s A_b^\dagger$ is an ordered product of rank-one operators $|k\rangle\langle b|_S \otimes \mathbb{K}$: the encodings $|0\rangle\langle u'_r|$ of the affected new holes, the patch terms $|w_i\rangle\langle w_i|$, and the decodings $|u_t\rangle\langle 0|$ of the affected old holes. Every such product is the contraction of the network whose vertices are the bras and the kets of the factors, kept distinct, and whose wires join the output of a ket at a site to the next bra containing that site; open input legs are those of bras at sites reached by no earlier ket, open output legs those of kets at sites met by no later bra, the legs of the product zero vectors are fixed to 0, and sites in no factor are passed through.² No wire joins a vertex to itself and every vertex tensor is normalized, so Lemma 6.2 applies with one group per vertex. An operator read off from a tensor on the open-leg groups is that tensor, reshaped as a matrix, tensored with the identity on the untouched sites, so its operator norm is at most the Hilbert–Schmidt norm of the tensor, with no factor depending on the site dimensions.³ Under the four conditions only the patch bras and kets have open legs, so there are at most $g = 2m$ groups; the open inputs of a patch bra have at most two old owners and the open outputs of a patch ket at most two new owners, all in the specified list. Hence every branch is approximated within $2m/\sqrt{k}$ by at most k^{2m} product terms with coefficients of modulus at most one, and for every $\delta > 0$ (for instance $\delta = L^{-a}$) the rescaled sum $M_\delta = (1 + \frac{1}{2}\delta)^{-1} \sum_\beta |a\rangle\langle b| \otimes (\sum_i c_{\beta,i} T_{\beta,i}) \otimes \mathbb{K}$ is a contraction with $\|M_\delta - M\| \leq \delta$.⁴ Each product term $T_{\beta,i}$ is an explicit tensor product of normalized vectors on the open output groups of the patch kets, normalized covectors on the open input groups of the patch bras, product zero vectors on the selected hole squares, and identities.

The last reading of the proof (lines 306–316), that each product term together with the tag basis vectors is the operator of a monomial allowed by Theorem 5.2, is also formalized. The registers of a frame are listed as the source has them, one raw register per site and one register per tag, each held by its owner, and their tensor product is identified with the canonical coordinates in which the rewrite acts.⁵ For a branch (a, s, b) and one unit vector on every open-leg group, the product term in the tag block, $|a\rangle\langle b| \otimes T \otimes \mathbb{K}$, is a scalar of modulus at most one times the operator of the following monomial from the registers of $\mathcal{F}_{\text{old}}$ to those of $\mathcal{F}_{\text{new}}$:

1. on the open input legs of each patch bra, the normalized covector of the

²TNLean.PEPS.SiteChain.chainOp_apply.

³TNLean.PEPS.SiteChain.norm_approxOp_le, TNLean.PEPS.SiteChain.exists_chainOp_truncation.

⁴TNLean.PEPS.EncodedFrame.SmallPatchRewrite.exists_branch_truncation, TNLean.PEPS.EncodedFrame.SmallPatchRewrite.exists_contractive_approx.

⁵TNLean.PEPS.EncodedFrame.layoutRegs, TNLean.PEPS.EncodedFrame.layoutIso, TNLean.PEPS.EncodedFrame.layoutIso_layoutVec.

term, contracted at the at most two old owners of these legs (a normalized pair effect, or a private map when there is one owner), the registers then being prepared in $|0\rangle$ at the same owners;

2. at every site of a selected patch or hole square, the bra $\langle 0|$ at the old owner and the ket $|0\rangle$ at the new owner; this imposes the zero bras of the old-hole decoders and transfers the reset raw register to its new owner;
3. on the open output legs of each patch ket, the bra $\langle 0|$ at the at most two new owners followed by the normalized vector of the term (a normalized pair source, or a private map);
4. the bra $\langle b|$ on the affected old tags and the ket $|a\rangle$ on the affected new tags, each at its tag owner.

Every other register is untouched; a site in no selected square keeps its owner by condition (ii). Each group block is placed on the registers of the frame by exchanges of tensor factors and unitaries grouping the registers of each owner. The monomial uses only the specified parties and has at most $2m$ normalized pair sources and at most $2m$ normalized pair effects, one of each per patch vertex with open legs; a vertex without open legs contributes a scalar of modulus at most one.⁶ Hence M_δ is a sum of allowed monomials on the registers of the frames, with coefficients of modulus at most one.⁷

The polynomial count is formalized as well. The formal statement gives at most Nk^{2m} monomials with $k = \lceil (4Nm/\delta)^2 \rceil + 1$, where N is the number of branches; when every affected hole and every patch has at most D cylinder terms, $N \leq D^{r_{\text{new}}+m+r_{\text{old}}}$ and the count is at most $D^E k'^{2m}$ with $E = r_{\text{new}} + m + r_{\text{old}}$ and $k' = \lceil (4D^E m/\delta)^2 \rceil + 1$.⁸ The source's bound $N \leq L^{C_1}$ (lines 331–332) and its total Nk^{g_0} (lines 341–342) use two standing bounds of the setting: (a) the projectors of every hole and every patch are sums of $\sum_j d_j$ cylinder projectors, with $\sum_j d_j$ bounded by one fixed power of L (`05-frames.tex`, lines 34–46, applied to the patch projectors at lines 254–256); and (b) m , r_{old} and r_{new} are bounded, by the hypotheses of the lemma (lines 189–192), and so is the specified list of parties of condition (iv) (line 204). With $\sum_j d_j \leq CL^c$, $m, r_{\text{old}}, r_{\text{new}} \leq b$, at most b specified parties, $L \geq 1$ and $\delta = L^{-a}$, the formal count is at most

$$(C+1)^{3b} (16b^2(C+1)^{6b} + 1)^{2b} L^{3bc+4b(3bc+\lceil a \rceil)}, \quad (1)$$

a polynomial in L whose coefficient and degree depend only on C , c , b and a , and the participating parties of all monomials lie in one fixed list of at most b parties, as the bounded change of Theorem 5.2 requires.⁹ When moreover $\|\Omega\| \leq 1$ and

⁶`TNLean.PEPS.EncodedFrame.exists_rankOneBlock_chain`, `TNLean.PEPS.SiteChain.termOp_eq_smul_prod`, `TNLean.PEPS.EncodedFrame.SmallPatchRewrite.exists_termChain`.

⁷`TNLean.PEPS.EncodedFrame.SmallPatchRewrite.exists_allowed_monomial_approx`.

⁸`TNLean.PEPS.EncodedFrame.SmallPatchRewrite.card_branch_le`, `TNLean.PEPS.EncodeFrame.SmallPatchRewrite.exists_allowed_monomial_approx_of_card_le`.

⁹`TNLean.PEPS.EncodedFrame.SmallPatchRewrite.exists_allowed_monomial_approx_polynomial`.

every additional patch projector and every affected old-hole projector has error at most the accuracy $\varepsilon = L^{-60}$ fixed for the frame construction (lines 21–25), the choice $a = 30$ gives the count (1) with $\lceil a \rceil = 30$ and the reference-vector error $\|M_{30}\Omega_{\mathcal{F}_{\text{old}}} - \Omega_{\mathcal{F}_{\text{new}}}\| \leq L^{-30} + 2bL^{-60} \leq (2b+1)L^{-20}$, the final assertion of the lemma.¹⁰ The bound on the participating parties cannot be derived from conditions (i)–(iii) alone. Every site of a selected hole square, including the sites of the inner hole, receives a zero bra at its old owner and a zero ket at its new owner (lines 258–268), and condition (iii) restricts the owners only outside the inner holes; an inner hole of side of order $\log L$ may have of order $\log^2 L$ distinct raw owners. The bound is the boundedness of the list in condition (iv), which the formal statement carries as the hypothesis that the list has at most b members. Bound (a) is a choice made in the source’s setting, so it enters the formal statement as a hypothesis on the cylinder data of the rewrite; that the choice can be made is Proposition 4.1, which is a separate result.

Two differences in form remain, and neither changes the statement. Each formal monomial has at most $2m$ normalized pair sources and $2m$ normalized pair effects, one of each per patch vertex with open legs, since the zero side of a two-owner block is also prepared and contracted as a pair vector; the source counts one per open-leg group, and Theorem 5.2 asks only for a bounded number. The expansion into monomials is stated in the canonical coordinates in which the tags of the untouched holes come first in both frames, as the source compares frames after the canonical identification of tag orderings (lines 74–75 and 84–85).

The formal frames record the pairwise disjointness of the physical outer footprints of holes rather than of the outer squares in the plane; the source condition implies the formal one, so this is a generalization. The rewrite is constructed in canonical coordinates in which the tags of the untouched holes come first in both frames. Since the encodings of holes with disjoint footprints commute, moving every tag to the new position of its hole is a relabelling of the tag basis that carries one encoding to the other; along it the canonical rewrite becomes a contraction between frames with arbitrary tag orderings, with the same reference error. This is the canonical identification of tag orderings of Definition 6.1, and it is formalized.

3 Resolution

Both assertions of Lemma 6.3 are formalized with the source’s hypotheses and the standing bounds of its setting. The scope restriction recorded in earlier versions of this note, first the reading of the product terms as allowed monomials and then the polynomial count, is removed.

¹⁰`TNLean.PEPS.EncodedFrame.SmallPatchRewrite.exists_allowed_monomial_approx_ofVec`.

References

- [Ope26] OpenAI. Polynomial PEPS approximation of gapped square-grid ground states, 2026. September 24, 2026 manuscript.