

Product-Reference Estimates and a Column on the Open Square

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Verdict: scope-restriction (open).

1 The source construction

Fix a site dimension $q \geq 2$, a bound $J > 0$ on the norms of the local interaction terms, and a full-system spectral-gap lower bound $\Delta > 0$. The positive constant C may vary between occurrences and depends only on q, J, Δ and explicitly fixed numerical powers, geometric templates, or tolerances. It is uniform in the lattice size L and in the Hamiltonian among those satisfying these fixed bounds [Ope26].

Section 8.1 of the polynomial PEPS approximation manuscript of September 24, 2026 [Ope26] (`07-assembly.tex`, lines 21–63) constructs an actual joint vector Ψ and a private reference vector Ξ such that, for the unit ground vector Ω ,

$$\|\Psi\| \leq 1, \quad 1 - CL^{-57} \leq \|\Xi\| \leq 1, \quad \|\Psi - \Omega \otimes \Xi\| \leq CL^{-17}. \quad (1)$$

It traces the private coordinates and applies distributed compression to this same joint vector. Lemma 8.1 (lines 67–94) extracts a nonzero column of the compressed operator on the party network. Section 8.3 then routes the ket network to the open square, preserving its contraction and bounding the resulting bond dimensions. The final constants must be uniform in the lattice size and Hamiltonian.

2 The conditional estimate

For arbitrary finite physical and private spaces, suppose

$$\|\Omega\| = 1, \quad \|\Psi\| \leq 1, \quad 1 - \delta \leq \|\Xi\| \leq 1, \quad \|\Psi - \Omega \otimes \Xi\| \leq \varepsilon. \quad (2)$$

The product marginal equals $\|\Xi\|^2 |\Omega\rangle\langle\Omega|$. The subnormalized pure-state estimate and contraction under partial trace therefore give

$$\left\| \text{tr}_E |\Psi\rangle\langle\Psi| - |\Omega\rangle\langle\Omega| \right\|_1 \leq 2\varepsilon + 1 - \|\Xi\|^2 \leq 2\varepsilon + 2\delta. \quad (3)$$

Here the last inequality uses $1 - s^2 \leq 2(1 - s) \leq 2\delta$ for $s = \|\Xi\| \leq 1$. It neither squares a possibly negative lower bound nor divides by s . The accepted generic result includes a zero private vector and an empty private coordinate set.¹

Now suppose an operator network with tensors A and edge dimensions D_e is already on the open $L \times L$ square. Write σ for its contracted operator, and assume

$$\|\sigma - \text{tr}_E |\Psi\rangle\langle\Psi|\|_1 \leq \kappa, \quad \alpha := \kappa + 2\varepsilon + 2\delta < 1. \quad (4)$$

The triangle inequality yields $\|\sigma - |\Omega\rangle\langle\Omega|\|_1 \leq \alpha$. Lemma 8.1 then supplies a product-basis index z and a real θ such that fixing the input of each tensor A to z gives

$$\Phi = \sigma|z\rangle \neq 0, \quad \left\| \frac{\Phi}{\|\Phi\|} - e^{i\theta}\Omega \right\| \leq 2\alpha. \quad (5)$$

Every edge keeps exactly its original dimension D_e . A zero dimension would make the contraction vanish, so $D_e > 0$; every bound $D_e \leq B$ is preserved. No positivity or Hermiticity of σ is assumed. The two error estimates use the same Ψ .²

3 Remaining construction

The conditional estimate does not construct Ψ , Ξ , their terminal physical and private coordinates, or the compressed network. To apply it to the source theorem, those objects must be identified with the actual protocol, and their three estimates must be proved with the same vector and constants uniform in the lattice size and Hamiltonian.

The statement above assumes an operator network already on the square, whereas the source fixes a column of the party network before routing it. The source order is now also available. For an operator party network on dyadic anchors satisfying the four routing properties of Lemma 8.2, an estimate $\|\sigma - |\Omega\rangle\langle\Omega|\|_1 \leq \eta < 1$ for its operator read at the folded anchors gives a nonzero column on the party graph, and routing that ket network gives a PEPS on the open square with the same contraction, positive bond dimensions at most $\max(D, 1)^\chi$, and normalized error at most 2η . The estimate may be supplied directly or through the product-reference bound above with $\eta = \alpha$. With link dimensions at most $C_1 L^k$, where $C_1 \geq 1$ and $k \geq 0$, the bond dimensions are at most $(C_1 L^k)^\chi$.³ The order difference is therefore resolved. The routing

¹The generic inequalities are in `QIClean/Analysis/ProductReferenceDensityError.lean`, in the companion library. In particular, `Matrix.rectangularTraceNorm_sub_pure_le_of_product_reference` includes the additional compressed-operator error.

²The two statements are `TNLean.PEPS.Approximation.Pinned.exists_fixInput_contractPEPS_of_product_reference` and `TNLean.PEPS.Approximation.Pinned.exists_peps_of_product_reference`, in `TNLean/PEPS/Approximation/ProductReferenceColumn.lean`.

³The statements are `TNLean.PEPS.Approximation.exists_squarePEPS_of_fixInput_dyadicRouting`, `TNLean.PEPS.Approximation.exists_squarePEPS_of_product_reference_dyadicRouting` and `TNLean.PEPS.Approximation.hasPEPSApproximation_of_fixInput_dyadicRouting`, in `TNLean/PEPS/Approximation/PartyColumnRouting.lean`.

properties remain hypotheses on the party network; they are recorded in the separate note on the routing lemma.

Finally, the output of Section 8.1 for large sizes, a compressed dyadic operator network within CL^{-5} of $|\Omega\rangle\langle\Omega|$ with polynomial link dimensions and constants chosen before L and the Hamiltonian, is stated as a proposition. That proposition implies the uniform approximation statement of Theorem 1.1, through the routed column and the exact finite-size construction.⁴ The proposition itself is not proved.

These are remaining implications in the formal construction, not errors in the source. The conditional statement is a separate mathematical consequence of supplied estimates; it is not Theorem 1.1. Resolving the restriction requires the actual protocol and compression estimates, with the routing properties of the actual party network, to prove the large-size proposition; the routing and the uniform large-size and finite-size arguments are already in place.⁵

References

- [Ope26] OpenAI. Polynomial PEPS approximation of gapped square-grid ground states, 2026. September 24, 2026 manuscript.

⁴The proposition is `TNLean.PEPS.Approximation.LargeSizeDyadicOperatorNetwork` and the implication is `TNLean.PEPS.Approximation.polynomialPEPSApproximation_of_largeSizeDyadicOperatorNetwork`, in `TNLean/PEPS/Approximation/GroundStateApproximation.lean`.

⁵The full target is tracked at <https://github.com/LionSR/TNLean/issues/8773>; distributed compression is tracked at <https://github.com/LionSR/TNLean/issues/8769>. The already established generic nonzero-column result is unchanged.