

Party Ownership in the Elimination of Normalized Pair Effects

2026-10-07

Verdict: scope-restriction (resolved).

1 The source lemma

Lemma 5.1 of the polynomial PEPS approximation manuscript of September 24, 2026 [Ope26] (`04-compression.tex`, lines 53–71) considers a contraction $G = \sum_{\xi} c_{\xi} M_{\xi}$ expanded into K allowed monomials, each with at most r pair effects. An allowed monomial (lines 32–35) is a composition of local contractions, preparations of normalized vectors shared between two participating parties, and contractions by normalized bras shared between two participating parties. For every $m \geq 1$ the lemma produces a gate $\tilde{G}_m$ and a normalized vector Γ_m with

$$\|\tilde{G}_m - G \otimes |\Gamma_m\rangle\| \leq \frac{r}{\sqrt{m}} \sum_{\xi} |c_{\xi}|, \quad (1)$$

and asserts four further properties of $\tilde{G}_m$: (i) it has an expansion using only local contractions and normalized pair sources; (ii) the expansion has at most Km^r monomials with absolute coefficient sum at most $\sum_{\xi} |c_{\xi}|$; (iii) its additional registers are owned by the original participating parties; (iv) all sources on one pair of parties may be combined into one normalized pair source.

2 The model without parties

The formal model writes a monomial with n effects as a chain $F_n \circ E_n \circ \dots \circ E_1 \circ F_0$, where each $E_i = \langle \eta_i | \otimes \mathbf{1}$ is a recorded normalized pair effect and each F_i is an arbitrary contraction. The model has no parties. The error bound, the normalization of Γ_m , the common additional output space of all branches, the count Km^r , the equality of the absolute coefficient sums, and the contractivity of every term of the expansion are proved.¹ For (iv), the formal statement is

¹The packaged statement is `TNLean.PEPS.PairEffect.pairEffectElimination`; the rescaling at a chosen stack length is `TNLean.PEPS.PairEffect.replaceGate_rescaled_stackLength`.

the norm identity $\|\eta \otimes \eta'\| = \|\eta\| \|\eta'\|$ for pair vectors regrouped on the tensor products of their half-spaces.

In that model clauses (i) and (iii) cannot be stated. Without parties, a local contraction cannot be told apart from an arbitrary contraction, and a pair effect is itself a contraction; a pair effect may therefore sit inside some F_i , and a predicate “source-only” formulated without parties would hold of every contraction.

3 Discharge

The monomials are now placed on a party layout. A register has an owner party and a Hilbert space, and the memory is the tensor product of the registers. A source-only word is a composition of local contractions, each acting on registers of a single party next to arbitrary other registers, of preparations of normalized vectors on two fresh registers of two distinct parties, and of reorderings of tensor factors. An allowed monomial is a chain of source-only words separated by pair effects, each effect contracting a register $\mathbb{C}^\alpha$ of a party p and a register $\mathbb{C}^\beta$ of a party $q \neq p$ with a normalized bra. Its operator is a chain of the form above, so the estimates already proved apply to it. Writing the effect on registers $\mathbb{C}^\alpha$ and $\mathbb{C}^\beta$ at the front of the list is a convention: an effect on finite-dimensional registers $\mathcal{U}$ of p and $\mathcal{V}$ of q in any position takes this form after local isometric equivalences $\mathcal{U} \cong \mathbb{C}^\alpha$ and $\mathcal{V} \cong \mathbb{C}^\beta$ and a reordering of tensor factors, all absorbed into the preceding source-only word without changing its norm or the coefficient of the monomial.

In the replaced monomial, the stack of an effect occurrence on (p, q) consists of m pair registers, the $\mathbb{C}^\alpha$ half of each owned by p and the $\mathbb{C}^\beta$ half owned by q . The insertion of the input at position k among $m - 1$ copies of η is a source-only word: pair sources of η are prepared in the other positions, so no register permutation is needed. Under the identification of the stack with $(\mathbb{C}^{\alpha \times \beta})^{\otimes m}$ this word is the insertion map I_k . Consequently:

- every term of the expansion of $\tilde{G}_m$, read on the output layout, is the operator of a source-only word with normalized sources and contractive local maps (clause (i));
- the output layout is the list of the stacks of all effect occurrences, followed by the output layout of G , and the stack of an effect on (p, q) consists of m registers of p alternating with m registers of q (clause (iii));
- the garbage vector Γ_m is prepared by normalized pair sources on the end-point parties;
- two pair sources on the same pair of parties, prepared one after the other and followed by merging the two registers of each party, are the single pair source of the regrouped vector, whose norm is the product of the norms.

The packaged statement of clauses (i)–(iii) and the bound is `TNLean.PEPS.PairEffect.partyPairEffectElimination`. Its hypotheses are those of the source lemma, without the contraction hypothesis on G , which the statement does not use. The participating parties of clause (iii) are read as the endpoint parties of the effects of the monomials. The combination of two adjacent sources is `TNLean.PEPS.PairEffect.eval_combineSources`.

4 Combining arbitrary source occurrences

The remaining combination clause is now proved for every allowed source-only composition F . Its finite list S records the actual occurrences of pair sources, with their endpoint parties, spaces and vectors. If P_S prepares these vectors before the input, there is an allowed composition B with no source preparations such that $F = B \circ P_S$. This identity is proved by induction on the composition. Moving a preparation before an earlier operation amounts to tensoring that operation with identities on the fresh registers. No bound on the number of occurrences is assumed.

A second induction combines sources with the same unordered pair of parties. When two sources have opposite endpoint order, a tensor exchange first aligns them. Their regrouped tensor product is normalized, and inverse local regroupings recover the original registers. Thus there is a list G with exactly one normalized source per participating unordered pair, and an allowed composition E with no sources, such that $P_S = E \circ P_G$. The set of pairs is unchanged. In particular $F = (B \circ E) \circ P_G$.

These statements are `TNLean.PEPS.PairEffect.Word.eval_eq_localPartCompPrepare`, `TNLean.PEPS.PairEffect.SourceInventory.exists_grouped`, and `TNLean.PEPS.PairEffect.Word.exists_grouped_source_preparation`. Applying them separately to every monomial in the replaced gate proves `TNLean.PEPS.PairEffect.partyPairEffectElimination_with_grouped_sources`. Every conclusion of Lemma 5.1 is retained, with exactly the same monomials, coefficients and common output registers. No adjacency, endpoint-order, Schmidt-rank or additional norm hypothesis is introduced.

This resolves the restriction recorded in this note. The subsequent collection into one contraction per party is proved by `TNLean.PEPS.PairEffect.Word.exists_tensorPartyMaps_parties`. For a prescribed finite gate party set, `TNLean.PEPS.PairEffect.Word.exists_prepared_tensorPartyMaps` combines it with the grouped preparation. These results use canonical identifications determined by the register owners and retain scalar local operations on empty memories. One-dimensional sources on unused pairs are provided by `TNLean.PEPS.PairEffect.Word.exists_complete_prepared_tensorPartyMaps`. The corresponding inventory completion retains every original source and recovers its preparation by allowed local operations. Common source spaces across different monomials (`04-compression.tex`, lines 253–267), followed by sampling and the remaining compression argument, still require further results.

References

- [Ope26] OpenAI. Polynomial PEPS approximation of gapped square-grid ground states, 2026. September 24, 2026 manuscript.