

# Bounded-Change Expansions of Births, Deaths and Exchanges

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**Verdict:** scope-restriction (resolved).

## 1 The source lemmas

Lemmas 6.5 and 6.6 of the polynomial PEPS approximation manuscript of September 24, 2026 (`05-frames.tex`, lines 396–570) change the ownership of registers of encoded frames.

*Lemma 6.5 (homogeneous birth and death).* In an encoded frame  $\mathcal{F}$ , let  $T$  be a set of raw sites owned by a party  $P_\circ$  and disjoint from every outer hole footprint, and put  $E = \{x : \text{the raw owner of } x \neq P_\circ\} \cup H^+(\mathcal{F})$  and  $U = \Lambda \setminus (T \cup E)$ . If  $I_\Omega(T : E) \leq L^{-60}$ , changing the owner of  $T$  to  $Q_\circ$ , with every hole, encoding and tag owner kept, “is a bounded change involving only  $P_\circ, Q_\circ$ , with reference-vector error at most  $L^{-30}$ ”; the reversed death obeys the same bound when the conditions hold in the frame after the death.

*Lemma 6.6 (two-sheet exchange).* For two encoded frames and a region  $Y$  such that every hole’s outer square lies on one side of  $\partial Y$ , with  $Z = \{x : \text{the two old raw owners of } x \text{ are not both } P_\circ\} \cup H^+(\mathcal{F}_1) \cup H^+(\mathcal{F}_2)$ ,  $T = Z \cap Y$ ,  $E = Z \setminus Y$  and  $U = \Lambda \setminus Z$ : if  $I_\Omega(T : E) \leq L^{-60}$ , exchanging the raw ownership assignments and the encoded holes inside  $Y$  “can be implemented by private contractions and register renaming, with reference-vector error at most  $4L^{-30}$ ”, using  $P_\circ$  and the owners of the renamed registers and tags, hence is a bounded change in the sense of Theorem 5.2.

In both lemmas the information bound is a hypothesis; it is supplied later by the manuscript’s proposition `prop:angular`.

## 2 What the formal statements cover

Both lemmas are formalized with exactly their source hypotheses, the information bound included.<sup>1</sup> The formal statements produce the canonical maps of

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<sup>1</sup>`TNLean.PEPS.EncodedFrame.Frame.birth`, `TNLean.PEPS.EncodedFrame.Frame.death`, `TNLean.PEPS.EncodedFrame.TwoSheetExchange.exchange` and `TNLean.PEPS.EncodedFrame.TwoSheetExchange.exchange_ofFrames`.

the proofs and prove:

1. for the birth and the death, that the canonical map  $B = \mathcal{V}^\dagger(|s\rangle\langle s|_{T\mathcal{B}_T} \otimes \text{id}_{E\mathcal{B}_E})\mathcal{V}$  is a Hermitian contraction, that it acts as the identity on  $E$  and on all tags, that it commutes with every hole encoder, and that the reference error is at most  $L^{-30}$ ; the map factors as  $B = (\mathcal{V}^\dagger S)(S^\dagger \mathcal{V})$ , where  $S$  inserts the single normalized vector  $s$  on  $T\mathcal{B}_T$ , and every register on which  $B$  acts lies outside  $E$ , so it is held by  $P_\circ$  before the birth and by  $P_\circ$  or  $Q_\circ$  after it; since the frames before and after differ only in raw owners, their encodings and reference vectors coincide, so the change of owner itself is carried entirely by the bounded-change clause below;
2. for the exchange, the exact identity  $\mathcal{R}K_{\text{in}} = K_{\text{out}}F_{Y \cap \Lambda}$  for the renaming  $\mathcal{R}$ , the identity  $D_U F_T = (\mathcal{V}^{\otimes 2})^\dagger F_{T\mathcal{B}_T} \mathcal{V}^{\otimes 2}$ , the identity  $CK_{\text{in}} = K_{\text{out}}D_U F_T$  for  $C = D_U F_A \mathcal{R}$ , contractivity of  $C$ , and the error bound  $4L^{-30}$  on the product of the two encoded reference vectors, which are not assumed normalized; the corrections  $D_U$  and  $F_A$  act on the registers of  $U$  alone, which are held by  $P_\circ$  on both sheets before and after the exchange, and the renaming keeps every register at its party.

The bounded-change clauses are formalized on the registers of the frames, as the source has them: one raw register  $\mathbb{C}^q$  per site and one register per tag, each held by its owner (`05-frames.tex`, lines 15–20 and Definition 6.1), with their tensor product identified with the canonical coordinates of the tags and the sheet in which the encodings act.<sup>2</sup> A monomial is a composition of contractions acting on registers of one party, normalized pair sources and normalized pair effects on registers of two named parties, and exchanges of tensor factors, as for Lemma 5.1.

1. The birth is an allowed monomial from the registers of  $\mathcal{F}$  to those of the frame in which  $T$  is owned by  $Q_\circ$ , using only  $P_\circ$  and  $Q_\circ$  and at most one normalized pair source, whose operator in canonical coordinates is  $\text{id}_{\text{tags}} \otimes B$ ; its reference-vector error, read on the registers, is at most  $L^{-30}$ . It moves the registers of  $T$  and of  $U$  in front by exchanges of tensor factors, groups each block into one register by a unitary at its owner  $P_\circ$ , applies the private contraction  $\langle s | (\text{id}_T \otimes V)$  at  $P_\circ$ , the pair source  $s$  on  $T$  held by  $Q_\circ$  and  $\mathcal{B}_T$  held by  $P_\circ$ , and the private contraction  $V^\dagger$  at  $P_\circ$ , and ungroups  $T$  at  $Q_\circ$  and  $U$  at  $P_\circ$ . For  $Q_\circ = P_\circ$  the middle step is the single private contraction  $K^\dagger K$ .<sup>3</sup>
2. The death is the same with one normalized pair effect  $\langle s |$  on  $T$  held by  $Q_\circ$  and  $\mathcal{B}_T$  held by  $P_\circ$  and no pair source.<sup>4</sup>
3. The exchange is the renaming  $\mathcal{R}$ , written as a word of exchanges of tensor factors that splits the tags of each frame into those of the holes outside and

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<sup>2</sup>`TNLean.PEPS.EncodedFrame.layoutRegs`, `TNLean.PEPS.EncodedFrame.layoutIso`, `TNLean.PEPS.EncodedFrame.Frame.regIs`, `TNLean.PEPS.EncodedFrame.Frame.regIso`.

<sup>3</sup>`TNLean.PEPS.EncodedFrame.Frame.birth_bounded`.

<sup>4</sup>`TNLean.PEPS.EncodedFrame.Frame.death_bounded`.

inside  $Y$ , moves the raw registers of the sites in  $Y$  next to the inside tags, exchanges these blocks between the sheets and merges back, followed by one private contraction at  $P_o$  on the raw registers of  $U$  of both sheets. It uses only  $P_o$  and no pair resource, its operator in canonical coordinates is  $C = D_U F_A \mathcal{R}$ , and its reference-vector error is at most  $4L^{-30}$ . For frames whose holes are listed in any order, the canonical identification of tag orderings is itself a word of exchanges of tensor factors, and the composite has operator  $CG$ .<sup>5</sup>

The renaming moves no register to another party. The source's assertion that the exchange uses  $P_o$  and the owners of the renamed registers and tags is satisfied with the stronger property that only  $P_o$  is used, since exchanges of tensor factors involve no party.

The formal frames record the outer footprints of holes by their physical samples. The source's condition that every outer square lies on one side of  $\partial Y$  enters as a classification of the holes as outside or inside  $Y$ , which is part of the data, together with its consequence for the samples: the sample of a hole classified as outside avoids  $Y$ , and that of a hole classified as inside lies in  $Y$ . A hole whose outer square has empty sample is compatible with both sides, so the side read from the plane must be supplied, and the formal statement accepts it. This is a representation of the source hypothesis, not an additional one.

### 3 Resolution

An earlier version of the formal statements proved the bounded-change clauses only on a layout in which the raw registers of  $T$ , and those of  $U$ , were grouped into one register each, followed by an arbitrary list of untouched registers, and identified the renaming of the exchange with an identification of tensor factors rather than a composition of exchanges. The list of the registers of a frame, one per site and per tag, and its identification with the canonical coordinates are now constructed, together with the words that move and group registers and the words that implement the renaming and the corrections. Composing the grouped monomials with these words gives the statements above, with the hypotheses of Lemmas 6.5 and 6.6. No restriction remains.

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<sup>5</sup>`TNLean.PEPS.EncodedFrame.TwoSheetExchange.exchange_bounded`, `TNLean.PEPS.EncodeFrame.TwoSheetExchange.exchange_ofFrames_bounded`.