PGWSVC08 Physical String Order versus Virtual-Boundary Nondecay

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Verdict: scope-restriction (open).

1 The source statement

The string-order paper [PGWS$^+$08] defines string order for a translationally invariant state Ψ by the existence of a local unitary $u \neq \nVdash$ and local operators x, y (hermitian without loss of generality) acting on the physical sites flanking the string, such that

$$\lim_{N\to\infty} |S_N(x,y,u,\Psi)| > 0, \qquad S_N(x,y,u,\Psi) := \langle\Psi|\, x \otimes u^{\otimes N} \otimes y \,|\Psi\rangle. \tag{\texttt{SOP}}$$

[1] Both the unitary and the endpoint operators are existentially quantified; the fixed-endpoint string order parameter of the Haldane-chain literature is the historical special case. For a pure finitely correlated state the correlator takes the transfer form

$$S_N(x,y,u,\Psi_\infty) = \operatorname{tr}\big[\Lambda\, \mathcal{E}_x\, \mathcal{E}_u^N\, \mathcal{E}_y(\nVdash)\big], \tag{\texttt{SOPMP}}$$

[2] where $\mathcal{E}_x, \mathcal{E}_y$ are the transfer maps twisted by the endpoint operators. The source's Theorem 1 then states: a pure finitely correlated state has string order if and only if there exist a unitary $\tilde{u} \neq \nVdash$, a matrix V, and indices n, m with

$$\mathcal{E}_{\tilde{u}}(V) = V, \qquad \operatorname{tr}(V\Lambda A_n A_m^\dagger) \neq 0. \tag{\texttt{Th1}}$$

[1]Source: [PGWS$^+$08, lines 112–122], local file `Papers/0802.0447/StringOrder-v10.tex`. A footnote of the source (lines 124–127) asserts, by citation to Verstraete–Martín-Delgado–Cirac, Phys. Rev. Lett. 92, 087201, that an "alternative but equivalent" definition gives the endpoints access to the virtual levels; the source does not prove this equivalence.

[2]Source: [PGWS$^+$08, lines 176–181].

[3] Immediately afterwards (lines 278–296) the source shows that the second condition can be dropped when x and y are products of observables on D^2 spins: the spans $S_m := \operatorname{span}\{A_{n_1} \cdots A_{n_m} A^\dagger_{m_m} \cdots A^\dagger_{m_1}\}$ grow strictly until they exhaust the full matrix algebra $M_D(\mathbb{C})$, so endpoint operators on sufficiently many physical sites reach every $D \times D$ matrix.

2 The restored peripheral phase

The source passage at lines 241–255 chooses right and left peripheral matrices V, Y for an eigenvalue $\lambda = e^{i\theta}$. Iterating the twisted transfer map contributes λ^N, so the leading asymptotic term is

$$S_N(x, y, u, \Psi_\infty) = e^{iN\theta} \operatorname{tr}\big[Y \mathcal{E}_y(\mathbb{1})\big] \operatorname{tr}\big[\Lambda \mathcal{E}_x(V)\big] + o(1). \tag{corrected asymptotic}$$

Lines 249–250 of the source display only the product of the two traces and omit $e^{iN\theta}$. The factor is required before the subsequent rephasing $\widetilde{u} = e^{-i\theta}u$. Its omission does not affect the absolute-value nonvanishing criterion because $|e^{iN\theta}| = 1$, but it does affect the complex asymptotic formula itself.

The blueprint therefore restores the factor in the proof of `thm:pgwsvc08_physical_string_order` in `blueprint/src/chapter/ch12_symmetry_string_order.tex`. The correlator is formalized as `MPSTensor.physicalStringOrderParam`; the source theorem and this asymptotic argument remain marked `notready`, so no Lean theorem is claimed for the restored expansion. The status is a deliberate local source correction, documented by the inline `Localfix` comment at the corrected display.

3 The formalized physical and virtual-boundary forms

The transfer-form physical correlator is now a formal definition:

$$S_N(x, y, u) = \operatorname{tr}\big[\Lambda\, \mathcal{E}_x\, \mathcal{E}_u^N\, \mathcal{E}_y(\mathbb{1})\big].$$

[4] The source-style existential predicate is also formalized: there exist a unitary $u \neq \mathbb{1}$, physical endpoint operators x, y, and a number $s > 0$ such that

[3]Source: [PGWS+08, lines 270–276]. The second condition is the realizability of the endpoint by the single-site operator $x = |n\rangle\langle m|$ with $y = x^\dagger \tilde{u}$, per the reduction at lines 257–268.

[4]`MPSTensor.physicalStringOrderParam`, `TNLean/MPS/Symmetry/StringOrderDefs.lean`.

$|S_N(x, y, u)| \to s$.[5]

The main theorems presently use virtual-boundary nondecay. Its virtual-boundary twist functional is $\mathrm{tr}(\Lambda X\, \mathcal{E}_u^L(Y))$ with arbitrary $X, Y \in M_D(\mathbb{C})$,[6] and the nondecay predicate is the existence of such X, Y together with a constant $c > 0$ bounding $\|\mathrm{tr}(\Lambda X\, \mathcal{E}_u^L(Y))\|$ from below at every length L.[7] The formal development proves the equivalence of virtual-boundary nondecay with virtual local covariance, and the universality of virtual-boundary nondecay for injective on-site-symmetric tensors with canonical normalisations.[8]

Virtual-boundary nondecay differs from physical endpoint string order in four ways.

1. *Physical versus virtual endpoints.* The source quantifies over physical operators x, y of bounded site support; the formal definition quantifies over arbitrary virtual matrices X, Y. A physical endpoint expands to a particular virtual matrix in the transfer picture, and the source's span argument (lines 278–296) provides the converse for injective tensors, but no formal statement connects the two notions.

2. *Limit versus uniform bound.* The source requires a positive limit $\lim_{N\to\infty} |S_N| > 0$; the formal definition requires a lower bound uniform in the length. A sequence with positive limit can vanish at finitely many small lengths and so fail the uniform bound for that witness pair.

3. *Left boundary form.* The source's left endpoint acts through the adjoint channel, $\mathcal{E}_x^\dagger(\Lambda)$; the formal expression multiplies, ΛX. These have the same expressive range when Λ is positive definite, as it is in all theorems concerned, but the formal expression is not the literal transfer form of the source.

4. *Fixed versus existential twist.* Physical endpoint string order quantifies existentially over the unitary u, whereas virtual-boundary nondecay fixes u as a parameter, in the style of the source's Theorem 2. The fixed-twist version is the finer statement, and an existential version is recoverable from it.

[5]`MPSTensor.HasPhysicalStringOrder`, same file.

[6]`MPSTensor.stringOrderBoundaryParam`, `TNLean/MPS/Symmetry/StringOrderDefs.lean`.

[7]`MPSTensor.HasStringOrder`, same file.

[8]`MPSTensor.stringOrder_iff_localSymmetry`, `MPSTensor.hasStringOrder_of_symmetric_injective`, `MPSTensor.hasStringOrder_iff_of_symmetric_injective`, `TNLean/MPS/Symmetry/StringOrder.lean`.

4 Verdict

Virtual-boundary nondecay is mathematically sound, and for the injective tensors that all concerned theorems treat, the existential physical-endpoint string-order and existential virtual-boundary criteria are equivalent by the source's own span argument. The physical correlator and the positive-limit source predicate are now formal definitions, but that equivalence is still the part with no formal counterpart: the development proves its nondecay theorems for virtual boundary witnesses only. Elimination requires proving the realizability of virtual boundary matrices by physical endpoint operators for injective tensors (a starting point is the single-site realization `MPSTensor.physRealize`[9]), and derive the source-faithful Theorem 1; tracked in GitHub issue #2660.

References

[PGWS+08] D. Pérez-García, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac. String order and symmetries in quantum spin lattices. *Physical Review Letters*, 100(16):167202, 2008.

[9] `TNLean/MPS/Chain/VirtualInsertion.lean`, currently not imported by the string-order files.