

Scalar Phases in the Physical String-Order Criterion

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Verdict: false-source (resolved).

1 The two literal assertions

Pérez-García, Wolf, Sanz, Verstraete, and Cirac define string order by

$$S_N(x, y, u) = \langle x \otimes u^{\otimes N} \otimes y \rangle, \quad \lim_{N \rightarrow \infty} |S_N(x, y, u)| > 0, \quad (1)$$

for some local unitary $u \neq \mathbb{1}$ and local endpoint operators x, y , which may be taken Hermitian. The exclusion is the literal identity matrix, not every scalar unitary. This is display SOP, lines 114–121 of the local source for arXiv:0802.0447 [PGWS⁺08].

For a pure finitely correlated state, Theorem 1 asserts that this property is equivalent to the existence of a unitary $\tilde{u} \neq \mathbb{1}$, a virtual matrix V , and physical letters n, m such that

$$\mathcal{E}_{\tilde{u}}(V) = V, \quad \text{tr}(V \Lambda A_n A_m^\dagger) \neq 0. \quad (2)$$

The source passage preceding this theorem replaces the original twist u by $\tilde{u} = \mu^{-1}u$, where μ is its peripheral eigenvalue. Although this preserves unitarity, it need not preserve $u \neq \mathbb{1}$: if $u = \mu\mathbb{1}$, the new twist is the identity. The relevant source passages are lines 257–264 and 270–275.¹

2 The literal string-order predicate is automatic

For every normalized state with positive physical dimension, choose $x = y = \mathbb{1}$ and $u = -\mathbb{1}$. Then

$$S_N(\mathbb{1}, \mathbb{1}, -\mathbb{1}) = (-1)^N, \quad |S_N(\mathbb{1}, \mathbb{1}, -\mathbb{1})| = 1. \quad (3)$$

¹Local source: `Papers/0802.0447/StringOrder-v10.tex`.

Both endpoints are Hermitian. For a canonical tensor this is the exact transfer calculation $\mathcal{E}_\mu \mathbb{1} = \mu \mathcal{E}$, $\mathcal{E}(\mathbb{1}) = \mathbb{1}$, and $\text{tr } \Lambda = 1$. Thus the literal definition supplies no nontriviality test on the state.

This observation alone does not disprove the existential conclusion in (2): a state could possess some other nonidentity fixed twist. The following example rules out all such twists.

3 A nondegenerate canonical example

Write $\sigma_x, \sigma_y, \sigma_z$ for the Pauli matrices. For $k = x, y, z$, set

$$\begin{aligned} U_0 &= \mathbb{1}, & U_k &= \frac{\mathbb{1} - i\sigma_k}{\sqrt{2}}, \\ A_j &= \sqrt{p_j} U_j, & \Lambda &= \frac{\mathbb{1}}{2}, \\ (p_0, p_x, p_y, p_z) &= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12} \right). \end{aligned} \quad (4)$$

Each U_j is unitary. Hence the transfer map is unital and trace preserving, and Λ is faithful, stationary, and has trace one. The four Kraus matrices are linearly independent: after removing their nonzero scalar factors, they are $\mathbb{1}, \mathbb{1} - i\sigma_x, \mathbb{1} - i\sigma_y, \mathbb{1} - i\sigma_z$, an invertible triangular change of the Pauli basis. Therefore they span $M_2(\mathbb{C})$. For each nonzero vector v , the vectors $A_j v$ span $\mathbb{C}^2$, so $\mathcal{E}(|v\rangle\langle v|)$ is positive definite. Full one-letter span consequently implies irreducibility and primitivity. In particular, every nonzero matrix X satisfying $\mathcal{E}(X) = \nu X$ with $|\nu| = 1$ has $\nu = 1$ and is scalar. The example therefore meets the source's canonical pure-FCS hypotheses; its bond space and physical alphabet are both nondegenerate.

On the traceless Pauli basis the transfer action is

$$T = \begin{pmatrix} 3/4 & -1/12 & 1/6 \\ 1/12 & 2/3 & -1/4 \\ -1/6 & 1/4 & 7/12 \end{pmatrix}. \quad (5)$$

Suppose a unitary u and nonzero Q satisfy $\mathcal{E}_u(Q) = Q$. The peripheral rigidity statement of the source's Lemma 1 supplies a unitary W and $|\mu| = 1$ with $\sum_j u_{ij} A_j = \mu W A_i W^\dagger$. Peripheral uniqueness forces $\mu = 1$. Consequently the transfer map commutes with conjugation by W .

Let R be the complex 3×3 matrix of this conjugation on the Pauli basis. Trace preservation of the bilinear form $(X, Y) \mapsto \frac{1}{2} \text{tr}(XY)$ gives $R^\top R = \mathbb{1}$.

Covariance gives $RT = TR$. Transpose this relation and use orthogonality to obtain $RT^\top = T^\top R$. Thus R commutes with

$$T + T^\top = \text{diag}(3/2, 4/3, 7/6).$$

Its diagonal entries are distinct, so R is diagonal. All off-diagonal entries of T are nonzero, so $RT = TR$ makes all three diagonal entries of R equal. Write $R = c\mathbf{1}$. Orthogonality gives $c^2 = 1$, whereas preservation of $\sigma_x\sigma_y = i\sigma_z$ gives $c^2 = c$. Therefore $c = 1$.

Conjugation by W fixes the identity and all Pauli matrices, hence every matrix. The intertwining equations now read $\sum_j u_{ij}A_j = A_i$. Linear independence of the physical letters yields $u = \mathbf{1}$. This contradicts the required nonidentity twist in (2); its nonzero trace coefficient already implies $Q \neq 0$. Yet (3) proves string order for this same tensor. The literal existential Theorem 1 is therefore false as printed.²

4 A phase-retaining correction

Keep the original physical twist u . Under the canonical pure-FCS hypotheses, physical endpoints with positive limiting magnitude exist if and only if there are a virtual matrix V , a phase $|\mu| = 1$, and physical letters n, m such that

$$\mathcal{E}_u(V) = \mu V, \quad \text{tr}(V \Lambda A_n A_m^\dagger) \neq 0. \quad (6)$$

To prove necessity, positive limiting magnitude excludes twisted spectral radius less than one. The unitary-twist bound and peripheral rigidity then supply a virtual unitary intertwiner. The physical endpoint coefficient criterion supplies the two letters. Conversely the nonzero coefficient makes V nonzero. Peripheral rigidity and uniqueness normalize its one-dimensional eigendirection to a unitary intertwiner, and the same endpoint criterion constructs physical endpoints.

5 The explicit projective convention

For physical string order, require the unitary twist to satisfy

$$u \notin \{c\mathbf{1} : c \in \mathbb{C}\}. \quad (7)$$

²The explicit declarations are `MPSTensor.stringPhaseTensor_canonical`, `MPSTensor.stringPhaseTensor_pure`, and `MPSTensor.stringPhaseTensor_literal_criterion_counterexample`, in `TNLean/MPS/Examples/StringOrderScalarPhase.lean`.

This is an explicit correction to the printed definition, rather than an unstated assumption on the source theorem. If $|\mu| = 1$, then u satisfies (7) exactly when $\mu^{-1}u$ does: an equality $\mu^{-1}u = c\mathbb{1}$ would give $u = (\mu c)\mathbb{1}$. Consequently the phase-retaining criterion implies the corrected global fixed-point criterion. Physical string order in this projective sense is equivalent to the existence of a nonscalar unitary $\tilde{u}$, a virtual matrix V , and letters n, m satisfying

$$\mathcal{E}_{\tilde{u}}(V) = V, \quad \text{tr}(V\Lambda A_n A_m^\dagger) \neq 0. \quad (8)$$

The reverse direction follows from (6) with $\mu = 1$.³

The four-letter example has no physical string order in this corrected sense, while its literal nonidentity-twist condition remains true. Both the AKLT and two-site cluster physical witnesses are nonscalar, so their physical endpoint conclusions are unchanged. Only this corrected physical predicate is retained. The literal condition appears in the explicit counterexample and scalar-twist calculation; it is not a second general predicate. The fixed-twist physical condition remains unchanged, as does virtual-boundary nondecay for fixed twists, including the identity.

The finite-support comparison and the D^2 Hermitian-product endpoint bound are established separately in `pgwsvc08_string_order_virtual_boundary.tex`, using `MPSTensor.pureCanonical_exists_isHermitian_physicalStringProductOrder`. The physical reduced-state symmetry theorem is a separate source assertion.

References

- [PGWS⁺08] D. Pérez-García, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac. String order and symmetries in quantum spin lattices. *Physical Review Letters*, 100(16):167202, 2008.

³The definition `MPSTensor.HasPhysicalStringOrder` now states the nonscalar condition. The global criterion is `MPSTensor.hasPhysicalStringOrder_iff_exists_fixed_letter`. The fixed-twist criterion is `MPSTensor.pureCanonical_exists_physicalStringOrderWith_iff_peripheral_letter`, both in `TNLean/MPS/Symmetry/PhysicalStringPhase.lean`.