

Global Phase Freedom in the PGVWC07 Quadratic Reconstruction System

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Verdict: false-source (open).

1 The printed assertions

Theorem 8 of Pérez-García, Verstraete, Wolf, and Cirac considers a translation-invariant state on a finite ring with unique open-boundary canonical form and successive reduced-state ranks bounded by D^2 . It forms a quadratic system from a window of D^4 matrices in the open-boundary canonical form. Its second conclusion states that every solution supplies a translation-invariant D -dimensional MPS representation of the state and is related to the canonical tensor by exact unitary conjugacy [PGVWC07, Theorem 8, lines 1154–1165].

Write $A = (A_i)_i$ for the canonical translation-invariant tensor and $B = (B_i)_i$ for the tensor component of a solution. The printed conclusion is therefore

$$V^{(N)}(B) = V^{(N)}(A), \tag{1}$$

$$A_i = UB_iU^\dagger \quad \text{for every } i \tag{2}$$

for some unitary U . Here $V^{(N)}(B)$ denotes the length- N trace-MPS vector generated by B .

The preceding uniqueness theorem makes the same exact-conjugacy assertion for two canonical translation-invariant representations of the same finite-ring state [PGVWC07, Theorem 7, lines 1002–1015]. Both assertions overlook the scalar phase freedom at fixed ring length.

2 Phase symmetry of the quadratic system

Use the zero-based notation of the blueprint and the formal quadratic-system structure. A window has matrices $A_i^{[j]}$ for $0 \leq j < m$. A solution consists of

matrices Y_j, Z_j and a tensor B satisfying

$$B_i \otimes \mathbf{1} = Y_j A_i^{[j]} Z_j, \quad (3)$$

$$Y_{j+1} Z_j = \mathbf{1} \quad (0 \leq j < m-1), \quad (4)$$

$$\sum_i B_i B_i^\dagger = \mathbf{1}. \quad (5)$$

This is the shifted indexing of the system printed immediately before Theorem 8 [PGVWC07, lines 1142–1152].

Let $\eta \in \mathbb{C}$ have unit modulus. Define

$$B_i^{(\eta)} := \eta B_i, \quad (6)$$

$$Y_j^{(\eta)} := \bar{\eta}^j Y_j, \quad (7)$$

$$Z_j^{(\eta)} := \eta^{j+1} Z_j. \quad (8)$$

Since $\bar{\eta}\eta = 1$, the gauge equation transforms as

$$Y_j^{(\eta)} A_i^{[j]} Z_j^{(\eta)} = \bar{\eta}^j \eta^{j+1} Y_j A_i^{[j]} Z_j \quad (9)$$

$$= \eta (B_i \otimes \mathbf{1}) \quad (10)$$

$$= B_i^{(\eta)} \otimes \mathbf{1}. \quad (11)$$

The internal inverse equation becomes

$$Y_{j+1}^{(\eta)} Z_j^{(\eta)} = \bar{\eta}^{j+1} \eta^{j+1} Y_{j+1} Z_j \quad (12)$$

$$= \mathbf{1}. \quad (13)$$

Finally, the normalization is unchanged:

$$\sum_i B_i^{(\eta)} (B_i^{(\eta)})^\dagger = \eta \bar{\eta} \sum_i B_i B_i^\dagger \quad (14)$$

$$= \mathbf{1}. \quad (15)$$

Thus every solution belongs to a full unit-modulus phase family of solutions. This freedom is present in the quadratic system itself and is not a virtual similarity gauge.

3 Effect on a finite-ring state

Every length- N coefficient is homogeneous of degree N in the local tensor. Therefore

$$V^{(N)}(B^{(\eta)}) = \eta^N V^{(N)}(B). \quad (16)$$

Suppose $V^{(N)}(B)$ is the prescribed nonzero state. If $\eta^N \neq 1$, then (16) shows that $B^{(\eta)}$ is another solution but does not represent that state. This contradicts the unrestricted representation claim (1).

If $\eta^N = 1$, then $B^{(\eta)}$ does represent the same finite-ring state. The exact-conjugacy claim can nevertheless fail. Take $d = D = 1$, let the single canonical matrix be

$$A_0 = 1, \quad (17)$$

and choose $N > 3$. This tensor represents a nonzero product state, satisfies condition C1 with onset $L_0 = 1$, has rank-one reduced states, and has no Schmidt-value degeneracy. Let $\eta \neq 1$ be an N th root of unity and set

$$B_0 = \eta A_0. \quad (18)$$

The phase construction above makes B the tensor component of a solution, and (16) gives $V^{(N)}(B) = V^{(N)}(A)$. In bond dimension one, unitary conjugation acts trivially:

$$UB_0U^\dagger = B_0 \quad (19)$$

$$\neq A_0. \quad (20)$$

Hence exact unitary conjugacy fails even after restricting to solutions that represent the same fixed-length state. This also gives a finite-ring phase counterexample to the exact conclusion printed in Theorem 7.

4 Scope of the endpoint-cut argument

The uniqueness argument compares open-boundary representations over the D^4 -site window. The complete dimension argument requires one intertwiner at each of the $D^4 + 1$ cuts and one relation across each of the D^4 sites. The off-by-one correction is recorded separately in `docs/paper-gaps/pgvwc07_intertwiner_chain_off_by_one.tex`.

The current quadratic-system structure produces only the internal-cut intertwiners. The two endpoint intertwiners must come from uniqueness of the open-boundary representation. That comparison is available only after the solution tensor is already known to represent the same state as the canonical tensor. It cannot prove that an arbitrary algebraic solution represents the state, even up to phase.

Once the same-state hypothesis supplies the endpoint cuts, the corrected chain argument gives a nonzero matrix R and a nonzero scalar x satisfying

$$RB_i = x^{-1}A_iR \quad \text{for every } i. \quad (21)$$

The normalization and fixed-point argument then gives

$$|x| = 1, \quad (22)$$

$$RR^\dagger = c\mathbb{1} \quad \text{for some } c > 0. \quad (23)$$

Thus $U = c^{-1/2}R$ is unitary and

$$A_i = xUB_iU^\dagger \quad \text{for every } i. \quad (24)$$

Because A and B generate the same nonzero length- N state, (24) implies

$$V^{(N)}(A) = x^N V^{(N)}(B) \quad (25)$$

$$= V^{(N)}(B), \quad (26)$$

$$x^N = 1. \quad (27)$$

The last conclusion uses that the state vector is nonzero.

5 Corrected theorem

The mathematically justified reconstruction statement keeps the existence clause of Theorem 8 and restricts its converse. Let B be the tensor of a solution whose length- N trace-MPS representation is the prescribed state. Then there are a unitary U and a scalar $\xi \in \mathbb{C}$ such that

$$|\xi| = 1, \quad (28)$$

$$\xi^N = 1, \quad (29)$$

$$A_i = \xi UB_iU^\dagger \quad \text{for every } i. \quad (30)$$

Equivalently, define the explicitly rephased tensor by

$$\widehat{B}_i := \xi B_i. \quad (31)$$

Then

$$V^{(N)}(\widehat{B}) = V^{(N)}(B), \quad (32)$$

$$A_i = U\widehat{B}_iU^\dagger \quad \text{for every } i. \quad (33)$$

The first relation is unit gauge-phase equivalence between B and A , while the second is exact gauge equivalence after the displayed scalar rephasing.¹

6 Open classification gap and verdict

The phase family proves that the unrestricted conclusion of Theorem 8 and the exact finite-ring conclusion of Theorem 7 are false as printed. The corrected same-state theorem above is the conclusion supported by the endpoint-cut argument.

A stronger classification remains open: the current quadratic equations have not been shown to imply that every solution is phase-gauge equivalent to the

¹The corresponding formal predicates are `MPSTensor.UnitGaugePhaseEquiv` and `MPSTensor.GaugeEquiv` in `TNLean/MPS/Defs`.

canonical tensor. The known phase-rescaled solutions do not show that they are the only spurious solutions. Proving such a result would require either a new argument deriving endpoint compatibility from the internal equations or an enlarged reconstruction system that includes endpoint data explicitly.

The blueprint therefore retains the algebraic reconstruction route but does not claim that an arbitrary solution represents the prescribed state. Its converse is restricted to same-state solutions and records the unavoidable N th-root phase before exact rephasing.

References

- [PGVWC07] D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac. Matrix product state representations. *Quantum Information & Computation*, 7(5):401–430, 2007.