

Scope of the Formal Periodic Decomposition Relative to PGVWC07

2026-08-12

Verdict: scope-restriction (open).

1 Source theorem

Pérez-García, Verstraete, Wolf, and Cirac prove the periodic decomposition theorem for translation-invariant MPS with periodic boundary conditions [PGVWC07, Theorem 5, lines 849–858]. The theorem starts with a TI state $\psi \in (\mathbb{C}^d)^{\otimes N}$ whose canonical TI MPS representation has a single block with $D \times D$ matrices A_i . Let

$$\mathcal{E}(X) = \sum_i A_i X A_i^\dagger.$$

If $\mathcal{E}$ has p eigenvalues of modulus 1, then:

1. if p divides N , the state ψ is a superposition of p p -periodic states, each with an MPS representation of bond dimension D ;
2. if p does not divide N , then $\psi = 0$.

The proof uses the peripheral spectral structure of the CP map from Fannes–Nachtergaele–Werner: there are orthogonal projectors P_k cyclically permuted by the matrices A_i , and inserting $\sum_k P_k$ into the trace splits ψ into the claimed p -periodic summands [PGVWC07, lines 860–880].

2 Formal boundary

TNLean proves the boundary-aware divisible-length decomposition in `MPSTensor.pgvlc07_periodic_stateVector_boundary_decomposition_of_dvd`, the positive-length explicit component-chain form in `MPSTensor.pgvlc07_periodic_stateVector_decomposition_of_dvd`, and the vanishing branch in `MPSTensor.pgvlc07_stateVector_eq_zero_of_not_dvd`. All three declarations use `MPSTensor.IsPeriodic`, which assumes left-canonical orientation, irreducibility, positive period, and an exact p -th-root peripheral spectrum. The

printed theorem instead starts from its one-block unital canonical representation and counts the peripheral eigenvalues of the associated transfer map. This normalized-orientation restriction remains tracked by issue #6232.

Let $A = (A^i)_{i=0}^{d-1}$ be the MPS matrix family. Let $(Q_u)_{u \in \mathbb{Z}/p\mathbb{Z}}$ be its cyclic projections and, for a word $\sigma = (i_0, \dots, i_{N-1})$, let

$$F_u(\sigma) = Q_u A^{i_0} Q_{u+1} A^{i_1} \dots Q_{u+N-1} A^{i_{N-1}} Q_{u+N}.$$

At every length, including $N = 0$, the formal decomposition is

$$\mathrm{tr}(A^{i_0} \dots A^{i_{N-1}}) = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} \mathrm{tr} F_u(\sigma).$$

For positive N , `MPSTensor.projectorComponentChain_coeff_eq_trace_cornerProd` identifies each boundary trace with the coefficient of the explicit periodic component chain. At $N = 0$, the boundary product is $F_u(\emptyset) = Q_u$, so the formula reads

$$\mathrm{tr}(\mathbb{1}) = \sum_{u \in \mathbb{Z}/p\mathbb{Z}} \mathrm{tr}(Q_u).$$

This must be distinguished from ordinary empty component-chain coefficients. An empty chain evaluates to $\mathbb{1}$ independently of u and therefore does not retain the boundary projector. The boundary-aware theorem closes the length-zero part of issue #6233 without changing the positive-length explicit component-chain statement.

3 Remaining scope restriction

To obtain a theorem with the printed hypotheses, prove that the paper's one-block unital canonical assumptions and peripheral-eigenvalue count imply the maintained periodicity hypotheses, after accounting for the adjoint change between unital and left-canonical orientations. Composing that result with the three declarations above would give the printed periodic decomposition and vanishing dichotomy. No further fixed-length coefficient calculation is required.

4 Verdict

Verdict: scope restriction. The positive-length restriction from issue #6233 is closed: the divisible-period branch now has a boundary-aware coefficient decomposition for every $N \geq 0$, while the explicit periodic component-chain theorem remains stated for $N > 0$, where its coefficients equal the boundary traces. The separate normalized-orientation restriction remains open under issue #6232, so the formal declarations are still normalized specializations of PGVWC07 Theorem 5 rather than a theorem from its literal one-block unital hypotheses.

References

- [PGVWC07] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Matrix product state representations. *Quantum Information and Computation*, 7(5–6):401–430, 2007.