The Intertwiner Count in the Quadratic Reconstruction Argument

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Verdict: local-correction (open).

1 The printed dependence step

The proof of Theorem 7 of Pérez-García, Verstraete, Wolf, and Cirac [PGVWC07] compares two translation-invariant matrix product representations through matrices acting on a D^2-dimensional doubled bond space. The local source is `Papers/quant-ph_0608197/MPSarchive.tex`, lines 1080–1095. It produces D^4 invertible matrices $W_1, \ldots, W_{D^4}$ satisfying

$$W_k(C_i \otimes I) = (B_i \otimes I)W_{k+1} \qquad (1 \leq k \leq D^4 - 1), \tag{1}$$

and then chooses an index n for which $W_1, \ldots, W_{n-1}$ are linearly independent but W_n belongs to their span.

The ambient matrix space is

$$M_{D^2}(\mathbb{C}), \qquad \dim_{\mathbb{C}} M_{D^2}(\mathbb{C}) = D^4. \tag{2}$$

Consequently, D^4 matrices need not be linearly dependent. The dependence asserted after (1) does not follow from the printed count.

2 The corrected cut count

A window of D^4 consecutive open-boundary matrices has D^4+1 cuts: one before the first site, one after the last site, and $D^4 - 1$ internal cuts. If a comparison of the two open-boundary representations supplies an intertwiner at every cut, then one obtains

$$W_0, W_1, \ldots, W_{D^4} \in M_{D^2}(\mathbb{C}), \tag{3}$$
$$W_k(C_i \otimes I) = (B_i \otimes I)W_{k+1} \qquad (0 \leq k \leq D^4 - 1). \tag{4}$$

The $D^4 + 1$ vectors in (3) are necessarily linearly dependent as a family. Since every W_k is invertible, $W_0 \neq 0$. Choose the first dependent vector. Then there is an index $1 \leq n \leq D^4$ such that $W_0, \ldots, W_{n-1}$ are linearly independent and W_n lies in their span. After shifting indices, this is exactly the dependence hypothesis required by the chain-combination lemma at source lines 1022–1049.

The current formal quadratic-system structure stores only gauges attached to the D^4 sites. It therefore constructs the $D^4 - 1$ internal-cut intertwiners and proves the $D^4 - 2$ relations between consecutive internal cuts. It has no fields representing the gauges at the two endpoint cuts. Those endpoint intertwiners must be derived from a future comparison theorem for the surrounding open-boundary representations. They are not consequences of the internal quadratic equations alone.

3 Verdict

The source's D^4-intertwiner count is off by one for the stated dimension argument. The local correction is to use the full chain of $D^4 + 1$ cut intertwiners around the D^4-site window. Once the two endpoint cuts and all D^4 adjacent relations are supplied, linear dependence and the subsequent combination argument are valid. The formal development currently proves only the internal chain and does not claim that this chain forces dependence, so the reconstruction capstone remains open.

References

[PGVWC07] D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac. Matrix product state representations. *Quantum Information & Computation*, 7(5):401–430, 2007.