

Half-Chain Spectrum: Canonical Orientation, Normalization, and Decay

2026-10-04

Verdict: scope-restriction (open).

1 Source and index convention

Theorem 6 of [PGVWC07], preceded by the calculation at lines 970–985 of `Papers/quant-ph_0608197/MPSarchive.tex`, identifies the limiting half-chain reduced spectrum with that of $\Lambda \otimes \Lambda$. The canonical convention is $\mathcal{E}_A(I) = I$ and $\mathcal{E}_A^*(\Lambda) = \Lambda$, where $\Lambda = \text{diag}(\lambda_\alpha) > 0$ and $\text{tr } \Lambda = 1$.

For $\mathcal{E}_A^L(X) \rightarrow \text{tr}(\Lambda X)I$, the source’s matrix element gives

$$\langle \alpha' | \mathcal{E}_A^L(|\beta'\rangle\langle\beta|) | \alpha \rangle \longrightarrow \lambda_\beta \delta_{\alpha\alpha'} \delta_{\beta\beta'}. \quad (1)$$

The printed display at line 979 instead places the weight on α . That placement is correct for the trace-preserving convention $\mathcal{E}_A^L(X) \rightarrow \text{tr}(X)\Lambda$. Taking adjoints of every Kraus matrix exchanges the conventions and swaps and transposes the Gram matrix. In both cases the limiting two-boundary product is $\lambda_\alpha \lambda_\beta$. The final unital theorem concerns the original tensor and its actual ring state; it does not replace the physical state by the adjoint-family state.

2 Finite-size normalization and zero eigenvalues

Let Ψ_L be the half-chain boundary family, $G_L = \Psi_L^\dagger \Psi_L$, and $W_L = S G_L^\top S$. The physical unnormalized reduced density is $\rho_L = \Psi_L W_L \Psi_L^\dagger$. The positive representative $H_L = \sqrt{G_L} W_L \sqrt{G_L}$ satisfies

$$X^{D^2} \chi_{\rho_L}(X) = X^{d^L} \chi_{H_L}(X), \quad (2)$$

$$Z_L := \text{tr } \rho_L = \text{tr } H_L = \sum_{\sigma, \tau} |V^{(2L)}(A)_{\sigma\tau}|^2. \quad (3)$$

Normalize both matrices by this actual Z_L , rather than assuming it is one at finite length. Convergence $H_L \rightarrow \Lambda \otimes \Lambda$ gives $Z_L \rightarrow 1$, hence eventual strict positivity. At a zero-norm length the normalized matrices are defined as zero.

List eigenvalues in decreasing order with multiplicity, then append zeros. Identity (2) identifies the two resulting sequences at every finite length, including $d^L < D^2$. The physical list is zero beyond its dimension and beyond D^2 .

3 Decay statement

The source states corrections of order $|\nu_2|^L$. Without semisimplicity at the sub-leading eigenvalues, powers of the complementary transfer map can carry polynomial factors from Jordan blocks. The geometric estimate used here chooses a radius strictly above the complementary spectral radius and below one, and bounds all Gram entries by Cr^L . This suffices for the stated spectral convergence.

Continuity of the positive square root applies on the entire positive cone, including singular limits. Weyl’s inequality then gives convergence of each ordered eigenvalue of H_L/Z_L ; the exact characteristic-polynomial identity transfers it to ρ_L/Z_L . No sharp quantitative rate for the square-root step, entropy convergence, or infinite-dimensional reduced operator is asserted.

4 Formal statements

The finite-size comparison and normalized spectral convergence are `MPSTensor.halfChainEigenvalues_eq_boundary` and `MPSTensor.tendsto_halfChainEigenvalues`. The normal-tensor unital statement is `MPSTensor.tendsto_halfChainEigenvalues_unital`; it assumes normality explicitly. Bare uniqueness of the peripheral eigenvalue does not imply irreducibility: the identity channel has peripheral spectrum $\{1\}$ but a non-scalar fixed-point space.

The source context includes all three one-block canonical conditions from Theorem 4, lines 742–759: unitality, a faithful dual fixed point, and scalar fixed points. Together with Condition C2 and its spectral-radius-one normalization at lines 954–963, these imply normality by `MPSTensor.isNormalTensor_of_pgvc07_canonical_c2`. The explicit source-hypothesis conclusion is `MPSTensor.tendsto_halfChainEigenvalues_of_pgvc07_canonical_c2`. For an explicitly one-block canonical tensor, no irreducibility or complementary spectral gap is assumed in addition to the three canonical conditions and normalized C2 hypothesis. The earlier normal-tensor result is a valid intermediate theorem, not by itself the formal statement from those source hypotheses.

The more general explicit-gap theorem allows singular nonnegative fixed points. Such points are outside the faithful canonical hypotheses of the source and are not asserted to be normal.

5 Remaining weighted multiblock scope

The canonical form in Theorem 4 is a weighted direct sum; the three canonical conditions apply separately to its blocks. Condition C2 does not alone

make the original representation a single faithful unital block. For example, let $0 < r < 1$ and $A_0 = \text{diag}(1, 0)$, $A_1 = \text{diag}(0, r)$. The transfer spectrum is $\{1, r^2, 0, 0\}$, with a simple peripheral eigenvalue one, but the canonical form has two weighted blocks and the ring vector is $|0^N\rangle + r^N|1^N\rangle$. The phrase “just one block” at source lines 960–963 therefore requires an asymptotic dominant-block interpretation rather than literal deletion of all subleading weighted blocks at finite length.

The new source-hypothesis corollary treats an explicitly one-block canonical tensor. It repairs the previously missing derivation of normality in that setting; it does not formalize the reduction from an arbitrary weighted canonical representation satisfying C2 to its dominant block. That reduction, including the disappearing subleading physical contribution and spectral normalization, remains open here. Accordingly, the unrestricted source theorem is not marked as completely formalized. The boundary-index and decay corrections in the preceding sections remain resolved independently of this representation-scope obligation.

References

- [PGVWC07] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Matrix product state representations. *Quantum Information and Computation*, 7(5–6):401–430, 2007.