The Direct-Sum Input from the Older MPS Representation Paper

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Verdict: scope-restriction (open).

1 Source statement

The finite-length direct-sum input used behind the block-injective canonical-form route is not proved in the 2016 MPDO source. The 2016 paper cites the older representation paper for the block-injective-after-blocking step [CPGSV16, lines 340–345]. The local source for the older argument is `Papers/quant-ph_0608197/MPSarchive.tex`.

That source first introduces the injectivity condition $C1$: for some L_0, the map

$$\Gamma_{L_0}(X) = \sum_{i_1,\ldots,i_{L_0}} \mathrm{tr}(XA_{i_1}\cdots A_{i_{L_0}})|i_1,\ldots,i_{L_0}\rangle$$

is injective. It immediately identifies this with the exact-length word products of length L_0 spanning the full matrix algebra [PGVWC07, lines 888–908].

The technical algebraic lemma then says that, for nonzero $D \times D$ matrices C, every matrix X is a finite sum

$$X = \sum_i R_i C S_i.$$

The proof reduces C to a matrix unit by elementary matrix operations and then reconstructs X from permuted copies of that matrix unit [PGVWC07, lines 1333–1344].

The next lemma is the direct-sum statement. Its local hypotheses are important: the tensor has $b \geq 2$ canonical blocks $A^1, \ldots, A^b$, each block satisfies $C1$, the common length is $L_0 = \max_j L_0^{A^j}$, the block sizes are ordered, and the block states are assumed pairwise different [PGVWC07, lines 1321–1330]. Under these hypotheses, if $L \geq 3(b-1)(L_0+1)$, then

$$\sum_{j=1}^{b} \mathcal{G}_L^{A^j}$$

is a direct sum [PGVWC07, lines 1346–1408]. The proof is by induction on the number of blocks. The base case uses $C1$ and the matrix-factorization lemma to show that a nontrivial linear dependence would force inclusion, then equality, of the two length-L_0+1 MPS subspaces, contradicting uniqueness of the associated injective parent-space characterization. The induction step isolates one block with a local projector, applies the induction hypothesis to the remaining blocks, and then uses $C1$ plus the factorization lemma to force the remaining coefficient to vanish.

2 MPS translation

For the non-periodic Fundamental Theorem route, the mathematical consequence is homogeneous and positive-length. After the canonical-form and basis-of-normal-tensors reduction, the selected representatives should admit one length at which their simultaneous word evaluations separate the direct-sum blocks. In trace-dual form, if

$$\sum_{j=1}^{g} \mathrm{tr}(\Delta_j A_j^w) = 0 \qquad \text{for every word } w \text{ of length } L,$$

then each Δ_j is zero. This is the finite-length direct-sum span input needed by the BNT pair-separation route.

This statement is not a consequence of the empty word and is not a cumulative span statement. The exact length L is part of the content. Therefore the formal proof should not replace the source input by a hypothesis that only says the cumulative pair algebra is full, nor by a theorem that uses the length-zero identity as padding.

3 Formal boundary

The matrix-factorization content of Lemma `lem1` from the source paper is now formalized in two steps.[1] The first says that, for any nonzero matrix C, the set of products $\{RCS : R, S \in M_D(\mathbb{C})\}$ spans $M_D(\mathbb{C})$. The second translates this into the relevant word-product form: if a tensor is L-block injective and $X \neq 0$, then $\mathrm{span}\{A_u X A_v : |u| = |v| = L\} = M_D(\mathbb{C})$.

The next part of the two-block direct-sum argument is also formalized. The three-block trace relation forces, in the size-ordering convention $D_B \le D_A$, an inclusion $\mathcal{G}_L^A \subseteq \mathcal{G}_L^B$ of length-L image spaces. Since block injectivity identifies $\dim \mathcal{G}_L^A = D_A^2$ and $\dim \mathcal{G}_L^B = D_B^2$, the inclusion and the size ordering together force $D_A = D_B$ and $\mathcal{G}_L^A = \mathcal{G}_L^B$.[2] The strict-size branch - if $D_B < D_A$, no nonzero

[1] `Matrix.span_range_mul_nonzero_mul_eq_top` for the abstract matrix-algebra form; `MPSTensor.span_range_evalWord_mul_nonzero_mul_evalWord_eq_top` for the word-product translation.

[2] `MPSTensor.bondDim_eq_and_groundSpace_eq_of_three_block_trace_relation_left_of_dim_ge`.

coefficient on the larger block can satisfy the homogeneous three-block trace relation - is formalized separately,[3] as is the conditional directness result: any external proof of the contradiction $\neg(D_A = D_B \wedge \mathcal{G}_L^A = \mathcal{G}_L^B)$ gives $\mathcal{G}_{3L}^A \cap \mathcal{G}_{3L}^B = \{0\}$.[4]

The equal-size uniqueness branch is isolated as well. For injective tensors, equal length-L image spaces force equality of the periodic MPV lines. A supplied distinct-MPV-line hypothesis therefore rules out the equal-size collapse.[5] The analytic source of distinct-MPV witnesses is formalized for irreducible trace-preserving blocks with normalized self-overlaps: if two blocks are not gauge-phase equivalent, then at arbitrarily large chain length their MPV states are not pointwise proportional.[6] Combining the uniqueness argument with the finite-dimensional image-space step gives: if any chain length $N \geq \max\{2, L\}$ witnesses non-proportionality, then $\mathcal{G}_{3L}^A \cap \mathcal{G}_{3L}^B = \{0\}$.[7] Zero intersection of image spaces at a fixed length, together with injectivity of the boundary-to-chain maps at that length, implies homogeneous pair trace separation.[8] The two branches are combined for all ordered pairs of distinct blocks of a BNT family at the common length $3L$, and the result is fed into the selector and product-span construction.[9]

The BNT conclusion is stated directly from explicit separated-block hypotheses, with the required finite-length injectivity hypotheses visible. These hypotheses may begin at an arbitrary positive common length L_0; positive-length propagation supplies the longer lengths used in the proof. The selector datum and the resulting finite direct-sum span are theorems rather than identity-padding assumptions.[10]

The sharp direct-sum interpolation does not concatenate a separate injective prefix. Full pair span at a common length S realizes an arbitrary value $(X, 0)$

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[3] `MPSTensor.not_three_block_trace_relation_left_of_isNBlkInjective_of_dim_gt`; `MPSTensor.pairTraceSeparatingAt_threeBlock_of_isNBlkInjective_of_dim_gt` for the trace-separation form; `MPSTensor.pairTraceSeparatingAt_threeBlock_of_isNBlkInjective_of_dim_ne` for the convention-free version.

[4] `MPSTensor.groundSpace_inf_eq_bot_of_not_bondDim_eq_and_groundSpace_eq_of_dim_ge`.

[5] `MPSTensor.not_bondDim_eq_and_groundSpace_eq_of_mpvSubmodule_ne`.

[6] `MPSTensor.exists_ge_not_forall_mpv_eq_mul_of_not_gaugePhaseEquiv_of_irreducible_TP`; `MPSTensor.exists_ge_not_forall_mpv_eq_mul_of_blocksNotGaugePhaseEquiv_of_irreducible_TP` for the BNT-family form.

[7] `MPSTensor.groundSpace_inf_eq_bot_of_exists_not_forall_mpv_eq_mul_of_dim_ge`; `MPSTensor.groundSpace_inf_eq_bot_of_blocksNotGaugePhaseEquiv_same_dim_of_dim_ge` for the BNT-separation packaging.

[8] `MPSTensor.pairTraceSeparatingAt_of_groundSpace_inf_eq_bot_of_isNBlkInjective`; and the combined forms `MPSTensor.pairTraceSeparatingAt_threeBlock_of_exists_not_forall_mpv_eq_mul_of_dim_ge`, `MPSTensor.pairTraceSeparatingAt_threeBlock_of_blocksNotGaugePhaseEquiv_same_dim_of_dim_ge`.

[9] `MPSTensor.forall_pairTraceSeparatingAt_threeBlock_of_blocksNotGaugePhaseEquiv`; `MPSTensor.exists_forall_pairTraceSeparatingAt_threeBlock_of_blocksNotGaugePhaseEquiv`.

[10] `MPSTensor.hasPairBlockSeparatingWords_threeBlock_of_blocksNotGaugePhaseEquiv`; `MPSTensor.propBlockInjective_of_blocksNotGaugePhaseEquiv_directSum`; `MPSTensor.wordTupleSpanTop_of_blocksNotGaugePhaseEquiv_directSum_selectors`; and their Condition-C1 variants in `TNLean/MPS/MPDO/BiCFDerivation/BNTDirectSum.lean`.

on a chosen block and one anchor block. Multiplication by $b-2$ identity–zero selectors removes the remaining blocks. Hence the full simultaneous span holds at length $(b-1)S$. For $S = 3(L_0+1)$ this is exactly the source length $3(b-1)(L_0+1)$.[11]

The downstream consumers of the direct-sum input are threefold. The exact-length span of a direct-sum block family and its two-block trace-dual form are expressed as predicates used throughout the pair route.[12] Theorems in the period-window module turn a positive homogeneous window of the pair-span predicate into homogeneous trace separation.[13] The BNT direct-sum theorems first turn pairwise non-equivalence into homogeneous three-block trace separation, and then construct selectors and a common exact-length simultaneous span.[14]

These results assume that distinct equal-dimensional blocks are not equivalent up to gauge and phase. PGVWC07 instead assumes only that the corresponding finite-ring component vectors are pairwise unequal. This inequality cannot imply gauge-phase separation. Gauge-phase equivalent blocks produce proportional finite-ring vectors, and distinct nonzero block weights can make them unequal while preserving proportionality. Thus the present formal results have a genuine scope restriction.

Eliminating the extra premise requires identifying, or proving from the source argument, the projective distinctness or nonproportionality condition actually needed by the interpolation step, and then deriving the formal pair-separation input from that condition. If the source hypothesis is read literally as mere inequality, the gauge-phase or nonproportionality premise must remain explicit. The current gap is not closed.

4 Conclusion

Under gauge-phase separation, the arbitrary-L_0 direct-sum interpolation needed for the BNT specialization is formalized at the exact source length $3(b-1)(L_0+1)$. The resulting span is homogeneous and positive-length, not cumulative. The length-zero identity is only the neutral base of a finite product over no remaining selectors. The case $b = 1$ is treated separately by ordinary block injectivity.

For a BNT canonical form of total bond dimension D, the common quantum Wielandt length is $L_0 = D^4$. Every representative has positive dimension and occurs at least once, so $b \leq D$. The formal theorem consequently gives a positive

[11] `MPSTensor.wordTupleSpanTop_mul_pred_of_forall_pairWordTupleSpanTop`; `MPSTensor.wordTupleSpanTop_threeBlock_mul_pred_of_blocksNotGaugePhaseEquiv_c1`.

[12] `WordTupleSpanTop`, `PairWordTupleSpanTop`, `PairTraceSeparatingAt`.

[13] `TNLean/MPS/MPDO/BiCFDerivation/PairHomogenization.lean`.

[14] `MPSTensor.forall_pairTraceSeparatingAt_threeBlock_of_blocksNotGaugePhaseEquiv`; `MPSTensor.hasPairBlockSeparatingWords_threeBlock_of_blocksNotGaugePhaseEquiv`; `MPSTensor.wordTupleSpanTop_of_blocksNotGaugePhaseEquiv_directSum_selectors` in `TNLean/MPS/MPDO/BiCFDerivation/BNTDirectSum.lean`; and `MPSTensor.wordTupleSpanTop_mul_pred_of_forall_pairWordTupleSpanTop`, `MPSTensor.hasBlockSelectorWords_of_pairBlockSeparatingWords` in `TNLean/MPS/MPDO/BiCFDerivation/Selectors.lean`.

simultaneous-span length at most

$$3(b-1)(D^4+1) \le 3(D-1)(D^4+1) \le 3D^5.$$

This proves the sharp block-separation input cited by the 2016 MPDO source under the additional gauge-phase separation premise. It does not close the input at the source's literal pairwise-inequality hypothesis.[15]

This conclusion concerns the finite-length direct-sum span. Other statements in the Fundamental Theorem that compare two canonical-form tensors still require their own reblocking and sector-data arguments; those are logically separate from the direct-sum input proved here.

5 Verdict

Verdict: scope restriction. The exact source length and direct-sum span are proved under gauge-phase separation, while PGVWC07 states only pairwise inequality of the finite-ring component vectors. That literal condition cannot imply the formal premise because unequal component vectors may still be proportional. Elimination requires identifying or proving the projective, nonproportional source condition used by the interpolation argument and deriving the formal pair-separation input from it. If the source is read as assuming only inequality, the additional separation premise must be retained. The sharp source input remains open at its printed hypothesis set.

References

[CPGSV16] J. Ignacio Cirac, David Pérez-García, Norbert Schuch, and Frank Verstraete. Matrix product density operators: Renormalization fixed points and boundary theories. Preprint; published version: Annals of Physics 378 (2017), 100–149, 2016.

[PGVWC07] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. Ignacio Cirac. Matrix product state representations. *Quantum Information and Computation*, 7(5–6):401–430, 2007.

[15] `MPSTensor.SectorDecomposition.basisCount_le_totalDim`; `MPSTensor.IsBNTCanonicalForm.exists_basis_wordTupleSpanTop_le_three_totalDim_pow_five`.