PEPS Gauge Uniqueness and Balanced Edge Scalars

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Verdict: false-source (resolved).

1 The assertion

The uniqueness clause in the Fundamental Theorem for injective PEPS on a simple graph appears in [MGRPG$^+$18, Theorem 2 and Section 3]. The argument first blocks the graph around a chosen edge to a three-site injective MPS. The one-dimensional fundamental theorem then assigns a gauge to that edge. Repeating this construction over all edges gives a family of edge gauges, and a local two-tensor comparison gives scalar proportionalities at the vertices.[1]

The theorem is then stated for PEPS on graphs with no double edges and no self-loops. It says that two injective PEPS generate the same state if and only if their tensors are related by local gauges. In short, the source says that the local gauges are "unique up to a multiplicative constant."[2] Interpreted as uniqueness up to one common scalar multiplying all edge gauges, this statement is too small for a general graph.

2 The corrected quotient

Let $G = (V, E)$ be a finite simple graph with a fixed orientation of each edge. For an edge e, a local gauge family assigns an invertible matrix X_e on the bond space of e. At the tail of e the endpoint action is X_e, and at the head of e it is X_e^{-1}.

A family of nonzero edge scalars $c = (c_e)_{e \in E}$ determines endpoint scalars

$$c_{v,e} = \begin{cases} c_e, & \text{if } v \text{ is the tail of } e, \\ c_e^{-1}, & \text{if } v \text{ is the head of } e. \end{cases}$$

[1] In the repository source, the edge-blocking step is at `Papers/1804.04964/paper_normal.tex:981--1009`; the assignment of edge gauges is at `Papers/1804.04964/paper_normal.tex:1037--1065`; the local scalar proportionality argument is at `Papers/1804.04964/paper_normal.tex:1066--1092` and `Papers/1804.04964/paper_normal.tex:1204--1207`.

[2] The quoted uniqueness phrase occurs in `Papers/1804.04964/paper_normal.tex:1207--1210`. For traceability, this note corresponds to GitHub issue #1045.

The family is vertex-balanced if

$$\prod_{e \ni v} c_{v,e} = 1 \qquad \text{for every } v \in V.$$

Two gauge families X and Y are equivalent modulo balanced edge scalars if there is such a vertex-balanced family c with

$$X_{v,e} = c_{v,e} Y_{v,e} \qquad \text{for every incident pair } (v, e),$$

where $X_{v,e}$ and $Y_{v,e}$ denote the oriented endpoint actions of the two families.

This is the graph-correct quotient. Rescaling the incident endpoint gauges at a vertex v by the factors $c_{v,e}$ multiplies the gauged local tensor at v by $\prod_{e \ni v} c_{v,e}$. If the scalar family is vertex-balanced, this product is 1 at every vertex, so all local gauged tensors are unchanged. Thus local gauge uniqueness can at best be uniqueness modulo these balanced edge scalars. The formal statement uses this quotient as the corrected uniqueness statement; the quotient relation is an equivalence relation on gauge families.[3]

3 The triangle witness

The obstruction already appears on the connected triangle. Let the vertices be $1 < 2 < 3$, orient every edge from the smaller endpoint to the larger endpoint, and let every bond dimension be 1. Then an edge gauge is just a nonzero complex number. Take any nonzero injective bond-dimension-one PEPS tensor A, and compare the identity gauge family with the scalar gauge family

$$c_{12} = t, \qquad c_{23} = t, \qquad c_{13} = t^{-1}, \qquad t \in \mathbb{C}^{\times}, \quad t^2 \neq 1.$$

At the three vertices the oriented products are

$$c_{12}c_{13} = 1, \qquad c_{12}^{-1}c_{23} = 1, \qquad c_{13}^{-1}c_{23}^{-1} = 1.$$

Hence the family is vertex-balanced. Applying this gauge family to A gives back the same local tensor at every vertex, so it is a second gauge family relating A to itself.

There is no single global scalar s with $c_{12} = c_{13} = c_{23} = s$. Indeed, $c_{12} = c_{13}$ would give $t = t^{-1}$, contradicting $t^2 \neq 1$. Therefore the global-scalar uniqueness interpretation is false even for a connected simple graph with one-dimensional bonds. The same example fits the balanced edge-scalar quotient exactly.[4]

[3] The formal definitions are `TNLean.PEPS.edgeScalarAt`, `TNLean.PEPS.IsVertexBalanced`, and `TNLean.PEPS.GaugeEquivModEdgeScalars`; the quotient relation is `TNLean.PEPS.GaugeEquivModEdgeScalars.setoid` in `TNLean/PEPS/FundamentalTheorem.lean`. The corresponding blueprint entries are in `blueprint/src/chapter/ch24_peps_ft.tex`.

[4] This is the counterexample recorded in `docs/counterexamples.md:127--135`. It originated in GitHub issue #762; the formal statement was corrected in PR #790.

4 Formal statement

The discrepancy is a statement-level divergence, not only an unfinished proof. The triangle example satisfies the graph-theoretic setting of the uniqueness clause and produces two genuine gauge families which differ by a balanced edge-scalar family but not by one global scalar. Consequently the source phrase must either be read with this quotient understood, or corrected to say uniqueness modulo balanced edge scalars.

The corrected quotient is now proved under the standing normal-PEPS convention that every bond space is nonzero. From equality of the gauged local tensors at each vertex, linear independence promotes the statement to equality of the products of incident edge-gauge entries over every configuration pair. Since each oriented endpoint gauge is invertible and, by the nonzero-bond convention, indexed by a nonempty type, the many-leg scalar extraction yields, at each vertex, nonzero proportionality constants with product 1. The lower-endpoint constant of each edge defines a single edge scalar; the inverse relation transports it to the upper endpoint, and invertibility makes each constant unique, so the per-vertex products identify with the oriented products of the edge scalars and are therefore 1.[5]

Scope restriction: nonzero bond spaces. The hypothesis $\dim_e > 0$ on every edge e is genuinely required, not a convenience. If a bond space is empty, the local virtual configurations at its two endpoints vanish, so the linear-independence hypothesis at those vertices is vacuous. A positive-dimensional bond adjacent to such an empty-bond endpoint is then left entirely unconstrained by the hypotheses: one can choose two gauge families that agree everywhere except on that bond, where they differ by no scalar, yet both satisfy all hypotheses vacuously. The conclusion fails for such a pair. The nonzero-bond convention is exactly the normal-PEPS assumption used in the existence direction (`hposA` in `TNLean.PEPS.fundamentalTheorem_PEPS`), so the uniqueness clause carries the same convention.

Thus the corrected mathematical statement is uniqueness modulo balanced edge scalars under nonzero bond spaces, and the triangle example refutes the smaller global-scalar quotient.

References

[MGRPG+18] András Molnár, José Garre-Rubio, David Pérez-García, Norbert Schuch, and J. Ignacio Cirac. Normal projected entangled pair states generating the same state. *New Journal of Physics*, 20(11):113017, 2018.

[5] For traceability, the declaration is `TNLean.PEPS.gauge_unique_mod_edge_scalars`; the many-leg extraction is `TNLean.PEPS.piProduct_forms_scalar` in `TNLean/PEPS/TensorFactorScalar.lean`. This point is recorded in GitHub issue #842; PR #879 reduced the formal statement to the scalar-ratio and global-reconciliation step, since completed.