# Injective PEPS Gauge Consistency Needs a Connectivity Hypothesis

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**Verdict:** false-source (resolved).

## 1 The assertion

The gauge-consistency step of the Fundamental Theorem for injective PEPS on a simple graph asserts that two vertex-injective tensors generating the same state are related by one global edge-gauge family. In the source this is the passage after the equation labelled `eq:inj_equal_edge`: blocking one vertex against its complement gives, for each vertex, a scalar proportionality $A_v = \lambda_v \widetilde{B}_v$, and the source then says "the constants $\lambda_v$ can be incorporated into the gauge transformations."[1]

The formal statement is `TNLean.PEPS.gaugeConsistency`: for all $v, \eta, \sigma$,

$$B_v(\eta)_\sigma = \mathrm{gaugeVertex}(A, X)_v(\eta)_\sigma,$$

for a single global gauge family $X$ on the edges. The headline `TNLean.PEPS.fundamentalTheorem_PEPS` reduces to this step.

## 2 The statement is false without connectivity

The reduction to scalars is correct, but the absorption of the vertex scalars $\lambda_v$ into the edge gauges is *not* possible on an arbitrary graph. The reason is structural. Each $\lambda_v$ multiplies the whole vertex tensor; absorbing it into the edge gauges requires nonzero edge scalars $s_e$ whose oriented product around each vertex equals $\lambda_v^{-1}$. The oriented incidence product of edge scalars at a vertex contributes $s_e$ at the tail and $s_e^{-1}$ at the head, so for every assignment of edge scalars the product of all vertex values is 1 on each connected component. The vertex scalars $\lambda_v$ are therefore absorbable only when their reciprocal family has product 1 on each component, which is forced by the state equality only for a connected graph (where there is a single component).

For a disconnected graph the per-component products need not be 1, and the exact single-gauge conclusion fails. The simplest witness is the empty graph.

[1] `Papers/1804.04964/paper_normal.tex:1207`.

**Witness (empty graph on two vertices).** Take $G = (\{0,1\}, \varnothing)$, physical dimension 1, and tensors

$$A_0 = 2, \quad A_1 = 3, \qquad B_0 = 6, \quad B_1 = 1.$$

Both are vertex injective (each vertex tensor is a single nonzero number). The states agree:

$$\langle A \rangle = A_0 \, A_1 = 6 = B_0 \, B_1 = \langle B \rangle.$$

All bond dimensions are vacuously positive (there are no edges). On the empty graph the only edge-gauge family is the empty one, and $\mathrm{gaugeVertex}(A, X)_v = A_v$ for every $v$, so the conclusion forces $A = B$. But $A_0 = 2 \neq 6 = B_0$. Hence the conclusion of `TNLean.PEPS.gaugeConsistency` fails on this input.

This construction has been checked in Lean: the negation of the conclusion of `TNLean.PEPS.gaugeConsistency` for this input is a theorem whose proof uses only the standard logical dependencies `propext`, `Classical.choice`, and `Quot.sound`. The same input refutes `TNLean.PEPS.fundamentalTheorem_PEPS`, since the empty edge gauge gives $\mathrm{GaugeEquiv}\, A\, B \iff A = B$.

# 3 The correct hypothesis

The source theorem is intended for connected PEPS; a PEPS on a disconnected graph is a product of independent states and is outside the scope of the local blocking argument. The faithful repair is to add a connectivity hypothesis to `TNLean.PEPS.gaugeConsistency` and `TNLean.PEPS.fundamentalTheorem_PEPS` (for example `G.Connected`, or `G.Preconnected` together with nonemptiness of $V$). Under connectivity the vertex scalars satisfy $\prod_v \lambda_v = 1$ (from the nonvanishing state equality), and a spanning-tree construction produces edge scalars realizing $\lambda_v^{-1}$ as the oriented incidence product at every vertex.

# 4 What is established

- *Per-vertex scalar.* The reduction to per-vertex scalars follows by combining the post-absorption insertion equality `TNLean.PEPS.post_absorption_edge_insertion_equality` with the four two-block injectivity facts (`TNLean.PEPS.isTwoBlockInjective_vertexTwoBlock`, `TNLean.PEPS.isTwoBlockInjective_complementTwoBlock` for $A$ and the reindexed $\widetilde{B}$), the bridge `TNLean.PEPS.sameTwoBlockInsertions_of_edgeInsertedCoeff_eq`, and `TNLean.PEPS.one_vertex_complement_comparison` yields, at every non-isolated vertex $v$, a nonzero $c_v$ with $A_v = c_v \, \widetilde{B}_v$.
- *Gauge inversion.* Inverting the absorbed gauge recovers $B_v = c_v^{-1} \, \mathrm{gaugeVertex}(A, Z^{-1})_v$, reducing the goal to the edge-scalar solve.
- *Gauge assembly (done).* The edge-scalar solve $\prod_{e \ni v} s_{v,e} = c_v^{-1}$ is solvable when $G$ is connected and $\prod_v c_v = 1$ (`TNLean.PEPS.exists_edg

eScalars_of_connected); folding the resulting edge scalars into the inverse of the absorbed gauge gives one global gauge family realizing $B_v = \mathrm{gaugeVertex}(A, X)_v$ at every vertex (TNLean.PEPS.perVertex_gauge_identity). The product-one condition $\prod_v c_v = 1$ is itself the cancellation TNLean.PEPS.prod_perVertexScalar_eq_one. With these, TNLean.PEPS.gaugeConsistency and the headline TNLean.PEPS.fundamentalTheorem_PEPS are reduced to a single isolated lemma.

- *State nonvanishing (proved).* The cancellation $\prod_v c_v = 1$ uses one nonzero state coefficient, $\exists \sigma,\ \langle A \rangle_\sigma \neq 0$ (TNLean.PEPS.exists_stateCoeff_ne_zero). For a vertex-injective PEPS with positive bond dimensions this is the source's standing assumption that an injective PEPS is a genuine nonzero state. The Lean derivation splits the state coefficient at one vertex against its complement (TNLean.PEPS.stateCoeff_eq_vertexComplement): the complement family is linearly independent (TNLean.PEPS.vertexComplementTensorInjective_of_isVertexInjective), hence has no zero member, and vertex injectivity at the chosen vertex then forces a nonzero coefficient; positive bond dimensions make the vertex-star configuration type nonempty. No induced-subgraph or leaf-peeling infrastructure is needed. This completes the reduction, so the headline TNLean.PEPS.fundamentalTheorem_PEPS is fully proved and axiom-clean.

This connectivity gap is distinct from the balanced-edge-scalar gap of docs/paper-gaps/peps_gauge_edge_scalars.tex: that note concerns the *uniqueness* clause (TNLean.PEPS.gauge_unique_mod_edge_scalars), where even a connected graph admits a non-global edge-scalar freedom; this note concerns the *existence* clause, where disconnectedness breaks the single global gauge outright.

# References