

A Two-Site Counterexample to Cyclic MPS Bond Uniformity

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Verdict: false-source (resolved).

1 The printed corollary

The Applications section of Ref. [MGRPG⁺18] considers a closed matrix product state whose local matrices have sizes $D_k \times D_{k+1}$. Its translational-invariance corollary asserts that translational invariance of the state and injectivity of the local tensors imply equality of all virtual dimensions and the existence of one repeated tensor. For this finite closed chain, the source writes translational invariance as equality with the one-site cyclic shift. Thus its bond-dimension conclusion is

$$D_1 = D_2 = \cdots = D_n. \quad (1)$$

The corollary itself does not state a lower bound on n .¹

The surrounding MPS section of Ref. [MGRPG⁺18] says that its Fundamental Theorem concerns chains with at least three sites, and the theorem invoked in the corollary is stated under that condition.² The omission in the corollary is therefore consequential rather than a harmless simplification: the literal two-site specialization is false.

2 A positive-dimensional two-site example

Let the physical space be $\mathbb{C}^2$, with standard basis e_0, e_1 , and take

$$D_1 = 1, \quad D_2 = 2, \quad (2)$$

$$A_1^i = e_i^\top \in \text{Mat}_{1 \times 2}(\mathbb{C}), \quad A_2^i = e_i \in \text{Mat}_{2 \times 1}(\mathbb{C}) \quad (i = 0, 1). \quad (3)$$

¹Local source: `Papers/1804.04964/paper_normal.tex:1803--1805`; the cyclic-shift equality is displayed at lines 1807–1827.

²Local source: `Papers/1804.04964/paper_normal.tex:145` and `Papers/1804.04964/paper_normal.tex:688--725`.

The first local tensor sends the two virtual basis vectors of $\mathbb{C}^1 \otimes \mathbb{C}^2$ to e_0, e_1 ; the second does the same from $\mathbb{C}^2 \otimes \mathbb{C}^1$. Both local tensor maps are therefore isomorphisms, in particular injective.

For physical indices $i, j \in \{0, 1\}$, the coefficient before and after the cyclic shift is

$$\text{tr}(A_1^i A_2^j) = e_i^\top e_j = \delta_{ij}, \quad (4)$$

$$\text{tr}(A_2^i A_1^j) = \text{tr}(e_i e_j^\top) = \delta_{ij}. \quad (5)$$

Equations (4) and (5) show that the two-site state is cyclically invariant. Every virtual space is nonzero and both local tensors are injective, while (2) contradicts (1).

The construction and the negation of the literal two-site statement have kernel-checked proofs.³

3 Corrected scope

The source proof compares the site-dependent chain with its cyclic shift by the injective MPS Fundamental Theorem, writes every A_i in terms of A_1 and the matrices L_i, R_i , and then telescopes these matrices to construct a repeated tensor B . That argument is available in the source’s stated regime $n \geq 3$ and does not apply to the example above. The corrected statement therefore retains the standing three-site hypothesis:

$$n \geq 3 \implies D_1 = D_2 = \dots = D_n, \quad (6)$$

together with the positive-dimensional convention for the virtual spaces. This is precisely the regime in which the graph-theoretic cycle has one edge for each bond and the cited Fundamental Theorem comparison is valid.

The complete corrected conclusion, including equality of all bond dimensions and the repeated injective tensor B , is formalized by `TNLean.PEPS.exists_constant_injectiveMPS_of_cyclicShiftInvariantState_per_bond`. It is recorded as complete in the Blueprint node `cor:exists_constant_injectiveMPS_of_cyclicShiftInvariantState_per_bond_pos`.

4 Verdict

The translational-invariance corollary of Ref. [MGRPG⁺18] is false as printed if it is read at two sites. The example (3) satisfies the nondegenerate source hypotheses and has a cyclically invariant state, but its two virtual dimensions are unequal. The corrected statement is the at-least-three-site result (6), in accordance with the standing scope and the Fundamental Theorem used by the source proof. The impossible unqualified reading is therefore a resolved false-source statement, not a separate open proof obligation.

³Formal declaration: `TNLean.PEPS.CycleMPSTwoSiteCounterexample.twoSiteBondUniformityStatement_false` in [TNLean/PEPS/CycleMPSTwoSiteCounterexample](#).

References

- [MGRPG⁺18] András Molnár, José Garre-Rubio, David Pérez-García, Norbert Schuch, and J. Ignacio Cirac. Normal projected entangled pair states generating the same state. *New Journal of Physics*, 20(11):113017, 2018.