

# Guide Properties in the Angular Separation of Dyadic Repaintings

2026-10-07; resolved 2026-10-08; interface labels, sampled guides  
and one ratio for all generic displacements 2026-10-08

**Verdict:** scope-restriction (resolved).

## 1 The source statement

Section 7 of the September 24, 2026 polynomial PEPS manuscript<sup>1</sup> repaints an aligned square  $S = (nr, ns) + [0, n]^2$  of the dyadic hierarchy from its old label  $A$  to its final label  $B$ . Along each edge it uses the normal ratio  $x = d/w(s)$  with  $w(s) = 10^{-3} \min(s, n - s)$ , and it runs a central birth, seven elementary births and deaths per edge, and one lens exchange per edge. Lemma 7.2 asserts that there is one constant  $a_0 > 0$  such that, for every birth or death among these operations and for every point  $y$  of the closure of its changed region,

$$\text{dist}(y, \overline{\{f \neq P_\circ\}}) \geq a_0 \min\{n, d_V(y)\}, \quad (1)$$

where  $f$  is the surrounding guide of the schedule,  $P_\circ$  the surrounding label, and  $d_V$  the distance from the marked points. For a lens exchange the point  $y$  ranges instead over the closure of the positions that are not common  $C$  on the two guides before the exchange, and the obstructing set is the boundary of the lens  $Y = \{-7 < x < -7/2\}$ :  $\text{dist}(y, \partial Y) \geq a_0 \min\{n, d_V(y)\}$ .

## 2 The earlier formal statement

Initially the schedule of guides was not constructed. The estimate (1) was proved with  $a_0 = 1/5000$  for every guide having the properties that the source's proof uses:

- for an edge birth or death, the guides before, after and surrounding the operation carry the source's normal words on the band  $\{-8 < x < 2\}$ , and the guides before and after agree off that band;

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<sup>1</sup><https://github.com/openai/math/blob/ad7f1241b42e322a6451854ab7e4b4c146bf78a/preprints/Polynomial-PEPS-approximation-of-gapped-square-grid-ground-states-September-24-2026/build/main.tex>, file 06-geometry.tex, lines 254–319.

- for the central birth, the surrounding guide equals  $A$  on the open square;
- for the lens exchange, the main guide equals  $C$  on the exterior band  $\{-8 < x < 0\}$ , and the auxiliary guide carries the auxiliary word on  $\{-8 < x < 2\}$  and equals  $C$  off it.

The square is placed at the origin. These are the declarations `TNLean.PEPS.Approximation.BandOperation.clearance`, `TNLean.PEPS.Approximation.centralBirth_clearance` and `TNLean.PEPS.Approximation.lensExchange_clearance`.

The source derives the listed properties at lines 227–241: the bands  $\{-8 < x < 2\}$  along distinct edges are disjoint, so the operations along one edge do not meet those along another, and the exterior part of a band stays in one neighboring square, which carries a constant label. The disjointness is proved as `TNLean.PEPS.Approximation.disjoint_edgeBand`. The formal statements therefore cover the source’s operations once the guides are built, and no hypothesis absent from the source’s argument is used. A translate of  $S$  only translates the coordinates, and sup distances are translation invariant.

### 3 Resolution

The schedule of guides is now constructed, and Lemma 7.2 is proved for it on every square of the hierarchy, with  $a_0 = 1/5000$ .

The starting guide of a repainting of  $S = [0, n]^2$  is any labelling of the plane that is  $A$  on the open square and carries one label  $C_e$  on the open exterior neighboring square across each edge  $e$ . The guide after the central birth is  $B$  on the central region. The guide after the edges of a list  $E$  have been completed replaces the band  $\{-8 < x < 2\}$  of each completed edge by the word  $C_e \mid B$  at 0. During the construction along an edge  $e$ , the main and auxiliary guides are the completed guide and the uniform guide  $C_e$  with the band  $\{-8 < x < 2\}$  of  $e$  replaced by the current normal word. These are the declarations `TNLean.PEPS.Approximation.RepaintingBaseline`, `TNLean.PEPS.Approximation.RepaintingBaseline.completedGuide`, `TNLean.PEPS.Approximation.RepaintingBaseline.mainGuide` and `TNLean.PEPS.Approximation.RepaintingBaseline.auxGuide`.

The derivation of the guide properties at lines 227–241 is proved: by the band disjointness, on the band of an edge not yet completed the completed guide carries  $C_e \mid A \mid B$  at 0,1 (`TNLean.PEPS.Approximation.RepaintingBaseline.completedGuide_eq_edgeStartWord`); the lens exchange produces the main word (`TNLean.PEPS.Approximation.RepaintingBaseline.lensExchange_mainGuide`); consecutive elementary operations on one sheet match; and after the four edges the open square carries  $B$  while every point outside the closed square keeps its starting label. The estimates are transported to the translate  $(nr, ns) + [0, n]^2$  of the square, whose starting guide is uniform on  $n$ -blocks, and stated in the source’s form with closures: `TNLean.PEPS.Approximation.blockRepainting_central`, `TNLean.PEPS.Approximation.blockRepai`

`nting_auxStep`, `TNLean.PEPS.Approximation.blockRepainting_mainStep` and `TNLean.PEPS.Approximation.blockRepainting_exchange`. Their only hypothesis is  $n > 0$ ; the block labels and the final label are arbitrary.

One convention remains, and it follows the source's own. A guide of the source is specified by its open polygonal chambers, and its labels are read at the lattice sites, the cell centers  $(i + \frac{1}{2}, j + \frac{1}{2})$ , after one small generic displacement (lines 62–80). A formal guide is a labelling of the whole plane, and the estimates are proved for the labellings that the schedule specifies at each operation; within one edge construction consecutive labellings match exactly. Twice the schedule passes to a labelling that agrees with the one reached only off finitely many curves. At the start of the construction along an edge  $e$ , the completed guide is replaced by the main guide with the starting word  $C_e \mid A \mid B$ , which agrees with it off the curves  $x = 0$  and  $x = 1$  of the band of  $e$  (`TNLean.PEPS.Approximation.RepaintingBaseline.completedGuide_eq_mainGuide_start`). At the end of a repainting, the final guide is replaced by the block guide of the next repainting, which agrees with it off the boundary of the repainted square (`TNLean.PEPS.Approximation.blockRepainting_final`).

These replacements matter for the formal changed regions: the set where two labellings differ is sensitive to their values on a curve, and if the labelling reached at the start of an edge were carried through the operations unchanged, the curve  $x = 1$  would enter the changed region of a main death and the estimate would fail there. The estimates are therefore not claimed for every labelling that agrees with the schedule's off these curves. The replacements do not matter for the source: the labellings represent the same partition into open chambers, and no lattice site lies on the curves. A cell center is never on the boundary of an aligned square of integer side, and never on the curves  $x = 0$  or  $x = 1$  of the band of one of its edges: there  $d$  is a half-odd integer, so  $d \neq 0$  and  $1000d$  is an integer, while  $1000w(s) = \min(s, n - s)$  is a half-odd integer (`TNLean.PEPS.Approximation.IsCellCenter.mem_openSquare_or_notMem_closedSquare`, `TNLean.PEPS.Approximation.IsCellCenter.notMem_edgeInterface`). Hence the labels at every lattice site are unchanged by both replacements (`TNLean.PEPS.Approximation.blockRepainting_start_site`, `TNLean.PEPS.Approximation.blockRepainting_final_site`). This is the source's sampling convention, not a hypothesis of the formal statements.

## 4 Labels on the interface curves

On an interface curve  $x = t$  of a band the formal words take the label for  $x \geq t$ . The source reads a guide on a boundary after one small generic displacement  $v$  into an incident chamber. The normal coordinate  $d$  points into the square, and the curves have slopes at most  $8/1000$  in the edge coordinates. So a displacement whose two components are not small relative to each other crosses to the side  $x > t$  along two edges of a square and to the side  $x < t$  along the other two, and the formal labellings and the source's sampled guides differ on the interface curves of two edges.

The estimates do not depend on this difference. Say that a labelling  $f'$  reads  $f$  after the displacement  $v$  if  $f'(p) = f(p + \varepsilon v)$  for every point  $p$  and all sufficiently small  $\varepsilon > 0$ . If  $f'_1, f'_2$  read  $f_1, f_2$  after the same  $v$  and  $f_1 = f_2$  near  $p$ , then  $f'_1(p) = f'_2(p)$ . Hence the set where  $f'_1 \neq f'_2$  lies in the closure of the set where  $f_1 \neq f_2$ , the set where  $f' \neq c$  lies in the closure of the set where  $f \neq c$ , and the same holds for the positions that are not common  $C$ . Every estimate of Lemma 7.2 is stated for these closures, so each passes from the formal labellings to every reading of them, for every displacement  $v$  (`TNLean.PEPS.Approximation.IsDisplacedReading.closure_ne_subset`, `TNLean.PEPS.Approximation.blockRepainting_central_of_reading`, `TNLean.PEPS.Approximation.blockRepainting_auxStep_of_reading`, `TNLean.PEPS.Approximation.blockRepainting_mainStep_of_reading`, `TNLean.PEPS.Approximation.blockRepainting_exchange_of_reading`).

Readings of the formal labellings exist and are unique, for every displacement. Along a ray  $p + \varepsilon v$  the coordinates, the band width and the normal coordinates agree for small  $\varepsilon > 0$  with affine functions of  $\varepsilon$ , whose signs are then eventually constant; the bands, the central region, the sup squares and the blocks are finite combinations of strict inequalities and floors of such functions, so every formal labelling, and every one homogenized on finitely many open squares, is eventually constant along every ray (`TNLean.PEPS.Approximation.isRayConstant_blockSchedule`, `TNLean.PEPS.Approximation.IsRayConstant.exists_isDisplacedReading`, `TNLean.PEPS.Approximation.IsDisplacedReading.unique`). The region of a label in a reading lies in the closure of its region in the labelling, so a true vertex of a reading is a true vertex of the labelling, and the outer holes of the readings of the temporary guides are disjoint (`TNLean.PEPS.Approximation.homogenize_reading_outerHoles_disjoint`).

## 5 The sampled guides are the readings

The source's sampled guides are identified with these readings. The open chamber of a label of a labelling  $f$  is the interior of the region of  $f$  with that label, and the guide sampled from the chambers of  $f$  after  $v$  takes at  $p$  the label of the chamber containing  $p + \varepsilon v$  for all small  $\varepsilon > 0$  (`TNLean.PEPS.Approximation.IsChamberSampling`). A sampled guide is a reading. This models the source's open polygonal chambers as the interiors of the label regions of the formal labellings. The model is faithful because the formal guides agree with the source's chamber descriptions on every open chamber, merging adjacent chambers that carry one label does not change the sampled label, and a label confined to boundaries is never sampled, as line 77 of the source requires. What is proved about the chambers is the next paragraph: the formal guides are constant off a locally finite family of lines.

Every guide of the schedule, the block guide, every guide of the repainting of a block, and each of these homogenized on finitely many open squares, is constant near every point off a set that lies, near every point,

in finitely many lines whose normals are  $(\pm t, 1000)$  or  $(1000, \pm t)$  with  $t$  in  $T = \{-8, -6, -5, -4, -3, -2, -1, 0, 1, 2\}$ ; the values  $-8$  and  $2$  bound only the region of a band that a word replaces. Indeed the interface curve  $x = t$  of a band is  $1000d = t \min(s, n - s)$ , a union of two segments; the edges, the grid lines and the sides of the treated squares are lines with  $t = 0$ ; the grid lines form a locally finite family; and a strict inequality between continuous functions, a bound on  $d/w(s)$  off  $w(s) = 0$ , and the integer part of a coordinate are constant off the sets where the two sides agree or the coordinate is a grid value (`TNLean.PEPS.Approximation.HasLineWalls`, `TNLean.PEPS.Approximation.hasLineWalls_schedule`). So the open chambers of a formal guide are the open sets on which it is constant, off a locally finite family of segments. A displacement  $v$  is generic if it is parallel to none of these lines, that is, if it avoids finitely many lines through the origin;  $(1, 1)$  is generic, and so is every nonzero multiple of a generic vector. A line not parallel to  $v$  meets the ray  $p + \varepsilon v$  in at most one point, so for generic  $v$  the ray leaves the walls at once and lies, for all small  $\varepsilon > 0$ , in one open chamber (`TNLean.PEPS.Approximation.IsLocallyInLines.eventually_notMem`). Hence for every generic  $v$  the sampled guide of every guide of the schedule exists, is unique, and equals its reading (`TNLean.PEPS.Approximation.isChamberSampling_schedule`, `TNLean.PEPS.Approximation.exists_isChamberSampling_blockRepainting`), and the same holds for the guides point treated about one grid corner (`TNLean.PEPS.Approximation.isChamberSampling_pointTreated`). The estimates of Lemma 7.2 hold for the sampled guides with  $a_0 = 1/5000$  (`TNLean.PEPS.Approximation.blockRepainting_central_of_chamberSampling`, `TNLean.PEPS.Approximation.blockRepainting_auxStep_of_chamberSampling`, `TNLean.PEPS.Approximation.blockRepainting_mainStep_of_chamberSampling`, `TNLean.PEPS.Approximation.blockRepainting_exchange_of_chamberSampling`). The estimates need no hypothesis beyond  $n > 0$ ; the genericity of the displacement, which the source assumes, is used only for the existence of the sampled guides.

The uniform footprint ratios of the point treatment and of the block guides also hold for the readings, with holes at the true vertices of the readings, and with one ratio  $\nu$  for all generic displacements. The source chooses  $K_0$ , then  $\epsilon_0$ , then  $\nu$ , and only afterwards the clearance  $a$  and the information scale  $D$  (lines 559–563), while the displacement is chosen after all the guides are specified (lines 73–77); so its  $\nu$  must serve every generic displacement. A reading takes only values of the labelling, and a reading of a relabelled similar image of a template is the same image of a reading of the template; the templates are eventually constant along rays, so their readings serve as templates. Each template is constant off walls on lines with normals in the finite set of the previous paragraph. If  $v$  and  $w$  lie strictly on the same side of each of these finitely many directions, then for every point  $p$  and all small  $\varepsilon > 0$  the segment from  $p + \varepsilon v$  to  $p + \varepsilon w$  crosses no wall: a wall through  $p$  separates neither  $v$  from  $w$ , and the other walls near  $p$  stay away from  $p$ . Along that segment the template is locally constant, hence constant, so its readings after  $v$  and after  $w$  agree (`TNLean.PEPS.Approximation.HasLineWalls.eventua`

lly\_eq\_of\_sameSide). Hence a template has at most one reading for each pattern of sides, and only finitely many readings after generic displacements (TNLean.PEPS.Approximation.HasLineWalls.finite\_genericReadings). The compactness argument applied to these finitely many readings gives one ratio for all generic displacements (TNLean.PEPS.Approximation.exists\_pointTreatment\_footprintRatio\_reading, TNLean.PEPS.Approximation.exists\_pointTreatment\_footprintRatio\_iUnion\_reading, TNLean.PEPS.Approximation.exists\_blockGuide\_footprintRatio\_reading, TNLean.PEPS.Approximation.exists\_directRepainting\_footprintRatio\_reading, TNLean.PEPS.Approximation.exists\_resizing\_footprintRatio\_reading, TNLean.PEPS.Approximation.exists\_resizing\_footprintRatio\_mixed\_reading).

Nothing of Lemma 7.2 or of its sampling convention remains open.