

The Lower Endpoint of Nachtergaele's Theorem 2.1(i)

2026-09-04

Verdict: false-source (resolved).

1 The source statement

Fix an increasing sequence of finite volumes $\Lambda_0 = \emptyset \subset \Lambda_1 \subset \Lambda_2 \subset \dots$ and local Hamiltonians H_Λ defined for every volume $\Lambda_n \setminus \Lambda_m$ with $m \leq n$. Nachtergaele's spectral-gap criterion assumes three conditions for one and the same finite length l , at lines 1030–1094 of the local source for Ref. [Nac96].

Condition C1 assumes, for a window parameter q , that there is a constant d_q with

$$0 \leq \sum_{n=q}^N H_{\Lambda_n \setminus \Lambda_{n-q}} \leq d_q H_{\Lambda_N}. \quad (1)$$

The constant does not depend on the volume: the source asserts (1) for every N with one d_q , and Theorem 2.1(i) is stated for every N with the same d_{l+1} . Condition C2 assumes that $\mathcal{G}_\Lambda = \ker H_\Lambda$ is a non-trivial subspace, writes G_Λ for the orthogonal projection onto $\mathcal{G}_\Lambda \otimes \mathcal{H}_{\Lambda_N \setminus \Lambda}$, and assumes a local gap $\gamma_q > 0$ with

$$H_{\Lambda_n \setminus \Lambda_{n-q}} \geq \gamma_q (\mathbf{1} - G_{\Lambda_n \setminus \Lambda_{n-q}}) \quad \text{for all } n \geq n_q. \quad (2)$$

With the conventions $G_{\Lambda_0} = \mathbf{1}$ and $G_{\Lambda_{N+1}} = 0$, the martingale differences

$$E_n = G_{\Lambda_n} - G_{\Lambda_{n+1}} \quad (3)$$

are mutually orthogonal projections summing to $\mathbf{1}$. Condition C3 assumes $\epsilon_l < 1/\sqrt{l+1}$ with

$$\|G_{\Lambda_{n+1} \setminus \Lambda_{n-l}} E_n\| \leq \epsilon_l \quad \text{for all } n \geq n_l. \quad (4)$$

Theorem 2.1(i), at lines 1119–1130, concludes that for any N and any ψ with $G_{\Lambda_N} \psi = 0$,

$$\langle \psi | H_{\Lambda_N} \psi \rangle \geq \frac{\gamma_{l+1}}{d_{l+1}} \left(1 - \epsilon_l \sqrt{l+1}\right)^2 \|\psi\|^2. \quad (5)$$

2 The point at issue

The printed proof, at lines 1195–1259, resolves ψ over the whole range

$$\psi = \sum_{n=0}^{N-1} E_n \psi, \quad \|\psi\|^2 = \sum_{n=0}^{N-1} \|E_n \psi\|^2, \quad (6)$$

derives for each index n and each pair of positive weights c_1, c_2 the estimate

$$\left(2 - c_1 - \frac{\epsilon_l^2}{c_2}\right) \|E_n \psi\|^2 - c_2 \sum_{m=n-l}^n \|E_m \psi\|^2 \leq \frac{1}{c_1 \gamma_{l+1}} \langle \psi | H_{\Lambda_{n+1} \setminus \Lambda_{n-l}} \psi \rangle, \quad (7)$$

and then sums it over the same full range.

Three separate hypotheses enter (7) and its summation, and each carries its own lower endpoint. The Hamiltonian on the right comes from estimating $\mathbf{1} - G_{\Lambda_{n+1} \setminus \Lambda_{n-l}}$ by C2 for the volume $\Lambda_{n+1} \setminus \Lambda_{n-l}$, that is by (2) at window $q = l + 1$ and index $n + 1$, which needs $n + 1 \geq n_{l+1}$. The coefficient ϵ_l^2/c_2 comes from C3 at index n , which needs $n \geq n_l$. The summation applies C1 at $q = l + 1$; reindexing (1) by $k = n + 1$ turns its left-hand sum into

$$\sum_{k=l+1}^N H_{\Lambda_k \setminus \Lambda_{k-l-1}} = \sum_{n=l}^{N-1} H_{\Lambda_{n+1} \setminus \Lambda_{n-l}}, \quad (8)$$

so C1 budgets exactly the local terms with $n \geq l$. The summed estimate is therefore available on

$$n_0 = \max\{l, n_l, n_{l+1} - 1\} \leq n < N, \quad (9)$$

and not below. What it yields there is

$$\kappa (1 - t)^2 P \leq \langle \psi | H_{\Lambda_N} \psi \rangle + \kappa (1 - t) t S, \quad (10)$$

where

$$\kappa = \frac{\gamma_{l+1}}{d_{l+1}}, \quad t = \epsilon_l \sqrt{l+1} \in [0, 1), \quad (11)$$

$$P = \sum_{n=n_0}^{N-1} \|E_n \psi\|^2, \quad S = \sum_{m=\max\{0, n_0-l\}}^{n_0-1} \|E_m \psi\|^2. \quad (12)$$

Passing from (10) to (5) requires $P = \|\psi\|^2$ and $S = 0$. The stated conventions $\Lambda_0 = \emptyset$, $G_{\Lambda_0} = \mathbf{1}$, and $G_{\Lambda_{N+1}} = 0$ do not make the early differences E_n vanish, and the source states no padding convention for the early local Hamiltonians. The question is whether C1, C2, and C3 nevertheless force the missing mass to vanish.

3 The printed statement fails

They do not. The following model satisfies C1, C2, and C3 with their printed lower endpoints, for every l and every N ; its parameter δ is chosen after the length l , and every choice below $1/(l+1)$ violates (5).

Let the sites be $x = 1, 2, \dots$ with $\mathcal{H}_x = \mathbb{C}^2$ and basis $|0\rangle, |1\rangle$, let $\Lambda_n = \{1, \dots, n\}$, fix a length l , and then fix δ with $0 < \delta < 1/(l+1)$. Take one-site interactions

$$h_1 = \delta |1\rangle\langle 1|_1, \quad h_x = |1\rangle\langle 1|_x \quad (x \geq 2), \quad (13)$$

and set $H_\Lambda = \sum_{x \in \Lambda} h_x$ for every volume Λ , so that every $H_{\Lambda_n \setminus \Lambda_m}$ is defined. Writing $P_x = |0\rangle\langle 0|_x$, the kernel of H_Λ is spanned by $|0\rangle^{\otimes \Lambda}$ and

$$G_\Lambda = \prod_{x \in \Lambda} P_x, \quad E_n = \left(\prod_{x \leq n} P_x \right) (\mathbf{1} - P_{n+1}) \quad (n < N). \quad (14)$$

These ground spaces are non-trivial, the projections G_Λ commute, and $G_{\Lambda_0} = \mathbf{1}$, so the martingale structure of the source is in place.

Condition C1 holds with $d_q = q$ for every q and every N : each h_x lies in $H_{\Lambda_n \setminus \Lambda_{n-q}}$ for at most q values of n in $\{q, \dots, N\}$, and all terms are non-negative, so

$$0 \leq \sum_{n=q}^N H_{\Lambda_n \setminus \Lambda_{n-q}} \leq q H_{\Lambda_N}. \quad (15)$$

Condition C2 holds with $\gamma_q = 1$ and $n_q = q+1$: for $n \geq q+1$ the volume $\Lambda_n \setminus \Lambda_{n-q} = \{n-q+1, \dots, n\}$ excludes site 1, so its Hamiltonian is a sum of commuting complementary projections and

$$H_{\Lambda_n \setminus \Lambda_{n-q}} = \sum_{x=n-q+1}^n (\mathbf{1} - P_x) \geq \mathbf{1} - \prod_{x=n-q+1}^n P_x = \mathbf{1} - G_{\Lambda_n \setminus \Lambda_{n-q}}. \quad (16)$$

The middle inequality is checked in the joint eigenbasis of the commuting P_x : both sides vanish where every P_x acts as the identity, and the left side is at least 1 elsewhere. Condition C3 holds with $\epsilon_l = 0$ for every l and every $n \geq 0$, since P_{n+1} occurs in $G_{\Lambda_{n+1} \setminus \Lambda_{n-l}}$ and

$$G_{\Lambda_{n+1} \setminus \Lambda_{n-l}} E_n = \left(\prod_{x \leq n} P_x \right) P_{n+1} (\mathbf{1} - P_{n+1}) = 0. \quad (17)$$

For $n < l$ the volume is read with $\Lambda_{n-l} = \Lambda_0 = \emptyset$ and the same computation applies. Thus C1, C2, and C3 hold for one and the same l , with

$$d_{l+1} = l+1, \quad \gamma_{l+1} = 1, \quad \epsilon_l = 0, \quad n_l = l+1. \quad (18)$$

The constants are the ones the source itself would supply: they are uniform in N , and $\epsilon_l = 0$ is admitted by C3, whose only requirement is $\epsilon_l < 1/\sqrt{l+1}$.

Now take

$$\psi = |1\rangle_1 \otimes |0\rangle^{\otimes(N-1)}. \quad (19)$$

Then $P_1\psi = 0$, hence $G_{\Lambda_N}\psi = 0$, and ψ is a unit vector with $H_{\Lambda_N}\psi = \delta\psi$. The printed conclusion (5) would give

$$\delta = \langle \psi | H_{\Lambda_N} \psi \rangle \geq \frac{\gamma_{l+1}}{d_{l+1}} (1-0)^2 = \frac{1}{l+1}, \quad (20)$$

which is false by the choice $\delta < 1/(l+1)$ made above. That choice depends on l but not on N , so the failure is not an artefact of small volumes: it persists for every $N \geq 1$ with the same l , the same constants, and the same threshold n_l .

The mechanism is exactly the one located in the previous section. Here $\psi = E_0\psi$, so the entire martingale mass sits at the index $n = 0$, which is below the threshold (9) for every l . Condition C2 fails at that index — the weakened bond h_1 has local gap δ , not $\gamma_{l+1} = 1$ — and this is precisely what its lower endpoint $n_{l+1} = l+2$ permits. The printed proof estimates the index regardless.

A formal instance of this model is verified in the Lean development. Every operator above is diagonal in the computational basis, so the span of the ground state $|0^N\rangle$ together with the single-excitation vectors $|0^m 1 0^{N-1-m}\rangle$, $0 \leq m < N$, is invariant under all of them, and the model restricts to that span, of dimension $N+1$, with E_m the projection onto the m -th excitation coordinate. Retaining the ground-state coordinate keeps every $\mathcal{G}_{\Lambda_n}$ non-trivial, as C2 requires; on the restricted space $\mathcal{G}_{\Lambda_N}$ is the line it spans, and ψ is orthogonal to it. Each ground space of the restricted model is again the kernel of the restricted Hamiltonian of its volume, and for $M > N$ the restricted H_{Λ_M} coincides with H_{Λ_N} , because the retained vectors carry no excitation beyond site N ; the whole family of volume Hamiltonians and the conventions $G_{\Lambda_0} = \mathbf{1}$ and $G_{\Lambda_{N+1}} = 0$ therefore survive the restriction. Conditions C2 and C3 are read at the volume Λ_N of the conclusion: their source ranges extend upwards without bound, but at an index $n \geq N$ the window $\Lambda_{n+1} \setminus \Lambda_{n-l}$ leaves Λ_N and its operators do not act on $\mathcal{H}_{\Lambda_N}$, which is why the printed proof sums over $0 \leq n < N$. The model satisfies them at every index above their thresholds, on the volume that index names, by (16) and (17). Reading them at $n = N$ as well is not an option the source leaves open, because it contradicts the convention $G_{\Lambda_{N+1}} = 0$: that convention makes $E_N = G_{\Lambda_N}$, which is non-zero because C2 asks the ground space to be non-trivial, so C3 at $n = N$ would bound $\|G_{\Lambda_{N+1} \setminus \Lambda_{N-l}} G_{\Lambda_N}\|$ by $\epsilon_l < 1$, while for the model above that operator is $\prod_{x \leq N+1} P_x$, of norm one. The restriction of (13) at $l = 0$ and $\delta = 1/2$ refutes the printed statement for every chain length $N \geq 1$.¹

¹Formal declarations: `FrustrationFree.UnrestrictedNachtergaeleEstimate` transcribes (5) with each condition on the range the source gives it, namely C1 on $l \leq n < M$, the range of its reindexed window sum (8), imposed with a single constant at every volume Λ_M against the corresponding member of the family of Hamiltonians, and C2 and C3 on (9); it also carries the

4 The minimal repair

Along the printed route the missing hypothesis is determined uniquely. Suppose (5) is to be deduced from (10). Subtracting the two inequalities, the deduction is available exactly when

$$\kappa(1-t)^2(\|\psi\|^2 - P) + \kappa(1-t)tS \leq 0. \quad (21)$$

Here $\kappa > 0$ and $1-t > 0$, and by (6) the deficit $\|\psi\|^2 - P$ equals $\sum_{n < n_0} \|E_n \psi\|^2$, so both summands in (21) are non-negative. Both must therefore vanish, and the vanishing of the first already implies the vanishing of the second, since S is a subsum of it. The minimal added hypothesis is thus

$$E_n \psi = 0 \quad \text{for all } 0 \leq n < n_0, \quad (22)$$

which, because $\sum_{n < n_0} E_n = \mathbf{1} - G_{\Lambda_{n_0}}$ and the E_n are mutually orthogonal projections, is equivalent to

$$G_{\Lambda_{n_0}} \psi = \psi. \quad (23)$$

Under (22) the conclusion (5) holds with the printed coefficient, with C2 and C3 assumed only on the source range (9) and C1 only on the range $l \leq n < M$ of its own window sum, and only at the two volumes Λ_{n_0} and Λ_N among the volumes at which the source asserts it.² The passage from C1 on $l \leq n < N$ to the sum over the threshold range $n_0 \leq n < N$ that the printed proof estimates is available because C1 at the volume Λ_{n_0} makes the omitted initial sum non-negative, while C2 makes each retained term non-negative. The earlier full-range formulation `FrustrationFree.NestedGroundProjections.energy_lower_bound_of_nachtergaele_c1_c3_full_range`, which assumes C1, C2, and C3 on every index $0 \leq n < N$, is the case $n_0 = 0$ of the threshold estimate, where (22) is vacuous.

The repair is a hypothesis on the pair (ψ, n_0) , not a strengthening of C1, C2, or C3, and it is what applications supply. In the nonwrapping open matrix-product-state parent Hamiltonian the early martingale differences are identically zero, so (22) holds for every ψ ; that specialization is recorded in `docs/paper-gaps/cpgsv21_martingale_overlap.tex`.

ordering $l \leq n_0$ of that threshold, the identification of each ground space with the kernel of its Hamiltonian and the non-triviality of the ground spaces that C2 requires, imposed for every window inside Λ_N because only the gap inequality of C2 carries an onset, and the conventions $G_{\Lambda_0} = \mathbf{1}$ and $G_{\Lambda_{N+1}} = 0$. Then `FrustrationFree.not_unrestrictedNachtergaeleEstimate` refutes it. In the formal instance $l = 0$, so C1 is supplied on the full range $0 \leq n < M$, where the local terms sum to H_{Λ_M} exactly at every volume.

²Formal declaration: `FrustrationFree.NestedGroundProjections.energy_lower_bound_of_nachtergaele_c1_c3_of_martingaleDifference_below_eq_zero`; the implication from (23) to (22) is `FrustrationFree.NestedGroundProjections.martingaleDifference_apply_eq_zero_of_lt`.

5 Conclusion

Theorem 2.1(i) of Ref. [Nac96] is false as printed. Its conditions C1, C2, and C3 carry lower endpoints; its proof estimates every martingale difference; and no convention stated in the source removes the difference between the two ranges. The one-site chain (13) with a weakened leftmost interaction satisfies all three conditions with constants uniform in N , and for any $\delta < 1/(l+1)$ it violates the printed bound by the factor $\delta(l+1)$.

The correct statement adds (22), the vanishing of the martingale differences below the threshold (9). That hypothesis is not merely sufficient: by (21) it is forced by the estimate the printed proof actually establishes. The intended applications, in which the initial volumes carry no ground-space defect, satisfy it.

References

- [Nac96] Bruno Nachtergaele. The spectral gap for some spin chains with discrete symmetry breaking. *Communications in Mathematical Physics*, 175(3):565–606, 1996.