

Ground-Space Subtraction in the Infinite-Volume Gap Argument

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Verdict: open-gap (wip).

1 The source conclusion

Nachtergaele’s gap theorem [Nac96] includes an infinite-volume conclusion. Its hypotheses include a positive finite-range interaction, equality of its zero-energy state face with the prescribed GVBS face, and equality of the open-chain kernel with the prescribed local support space at every sufficiently large length. The interaction range need not exceed a separately supplied injectivity length. For each pure GVBS ground state η and local observable X with $\eta(X) = 0$, the uniform finite-volume gap $\gamma > 0$ also satisfies

$$\lim_{M \rightarrow -\infty, N \rightarrow +\infty} \eta(X^*[H_{[M,N]}, X]) \geq \gamma \eta(X^*X). \quad (1)$$

The local source is `References/cond-mat_9410110/main.tex`, lines 933–947. Its final proof, at lines 2649–2675, uses convergence of finite-volume pure ground-state expectations to those of one pure infinite-volume sector.

2 The finite-volume estimate

Let $H \geq 0$ act on a finite-dimensional Hilbert space, let $K = \ker H$, and suppose $H \geq \gamma P_{K^\perp}$. For every vector v , orthogonal decomposition gives

$$\langle Hv, v \rangle \geq \gamma(\|v\|^2 - \|P_K v\|^2). \quad (2)$$

If $\psi \in K$, then $\langle \psi, X^*[H, X]\psi \rangle = \langle HX\psi, X\psi \rangle$. Consequently

$$\langle \psi, X^*[H, X]\psi \rangle \geq \gamma(\|X\psi\|^2 - \|P_K X\psi\|^2). \quad (3)$$

These assertions are proved in `TNLean/MPS/ParentHamiltonian/Martingale/GroundExpectationGap.lean`.

The projection in (3) is onto the entire ground space. For a normalized ground vector ψ , the condition $\langle \psi, X\psi \rangle = 0$ removes the component along ψ ; it does not remove components along other vectors in K . Thus scalar centering alone does not justify replacing the projection term by zero in a degenerate finite volume.

3 The proved limiting implication

Consider Hilbert spaces and operators indexed by expanding volumes n . Suppose their Hamiltonians have the same gap γ for all sufficiently large n , and let $\psi_n \in \ker H_n$. If

$$\|X_n \psi_n\|^2 \longrightarrow a, \quad (4)$$

$$\|P_{\ker H_n} X_n \psi_n\|^2 \longrightarrow 0, \quad (5)$$

$$\langle \psi_n, X_n^*[H_n, X_n]\psi_n \rangle \longrightarrow e, \quad (6)$$

then (3) implies $e \geq \gamma a$. The formal limiting assertion permits the Hilbert space to vary with n . Its projection-decay hypothesis is explicit; it is not counted as a formalization of (1).

When $\ker H_n = \mathbb{C}\psi_n$ and $\|\psi_n\| = 1$ for all sufficiently large n , one has

$$\|P_{\ker H_n} X_n \psi_n\|^2 = |\langle \psi_n, X_n \psi_n \rangle|^2. \quad (7)$$

Thus convergence of the scalar mean to zero suffices in this case. The corresponding finite-volume estimate and limiting implication are also proved in the cited module. Finite-volume uniqueness is an explicit hypothesis; it is not inferred from purity of the limiting state.

4 Faithful generating data and finite supports

Let A be a tensor of positive bond dimension and let $\rho > 0$. For a k -site matrix X , put

$$\eta_{A,\rho}(X) = \frac{1}{\text{tr } \rho} \sum_{\sigma, \tau} X_{\sigma\tau} \text{tr}(A^\tau \rho (A^\sigma)^*).$$

No stationarity or normalization is required for the following equivalence:

$$\eta_{A,\rho}(X^* X) = 0 \iff G_A(k) \subseteq \ker X.$$

Indeed, writing $C_b = \sum_\sigma X_{b\sigma} A^\sigma$, one has

$$(\text{tr } \rho) \eta_{A,\rho}(X^* X) = \text{tr} \left(\rho \sum_b C_b^* C_b \right).$$

Faithfulness of ρ makes this trace zero precisely when every C_b is zero. The identity $(X\Gamma_k(Y))_b = \text{tr}(C_b Y)$ and the nondegeneracy of the trace pairing finish the argument. This is proved in `TNLean/MPS/ParentHamiltonian/LocalFaithfulSupport.lean`. The argument applies also at $k = 0$.

If $\sum_i A_i^* A_i = I$, the functional is positive and normalized. It therefore has a positive representing matrix σ_k with trace one:

$$\eta_{A,\rho}(X) = \text{tr}(\sigma_k X), \quad \text{ran } \sigma_k = G_A(k).$$

The range equality follows by comparing the vanishing conditions for $\eta_{A,\rho}(X^* X)$ and $\text{tr}(\sigma_k X^* X)$, then testing the orthogonal-complement projections of the two subspaces. The result is proved in `TNLean/MPS/ParentHamiltonian/LocalDensitySupport.lean`. Stationarity is unnecessary for these finite densities. Without stationarity the conclusion does not assert consistency across intervals.

Faithfulness of ρ can be removed for normalized stationary data. Let $\sum_i A_i^* A_i = I$, let $\rho \geq 0$ have nonzero trace, and let $\sum_i A_i \rho A_i^* = \rho$. The support projection P of ρ satisfies $(I - P)A_i P = 0$. Choose an isometry V with $VV^* = P$ and put $B_i = V^* A_i V$ and $\sigma = V^* \rho V$. Then $A_i V = V B_i$, $\sigma > 0$, and $\eta_{A,\rho} = \eta_{B,\sigma}$ on every interval. Moreover $G_B(k)$ is the image of the corner $PM_D P$ under the boundary map of A . Hence $\eta_{A,\rho}(X^* X) = 0$ exactly when X annihilates this corner image, and when A is left canonical the density σ_k has this corner image as its range. These statements are proved in `TNLean/MPS/ParentHamiltonian/StationaryLocalSupport.lean`.

The remaining step is the passage from an arbitrary GVBS boundary limit, Equation (1.1) of Ref. [Nac96], to normalized stationary data. Nachtergaele takes it from Fannes–Nachtergaele–Werner [Nac96], local source lines 1411–1414, and the faithful minimal data used at lines 1724–1738 also rely on the minimality condition of lines 1457–1467. That passage is not formalized here; the support results therefore start from normalized stationary generating data. Removing this restriction requires formalizing the cited normalization and minimality results from a general presentation. Likewise, results stated for a primitive tensor with canonical data, for a normalized periodic tensor, or for a finite periodic family take that tensor presentation as given. Deriving such a presentation from a general GVBS boundary-limit presentation is part of the same unformalized passage, and the same elimination plan applies.

5 The primitive-sector quasi-local inequality

For one normalized primitive tensor with a positive-definite invariant matrix, the tensor-specific projection decay is now proved. The exact boundary compression of an interior observable is the reshuffling of $E_A^\ell E_X E_A^r$. When both free intervals tend to infinity, this converges to $\eta_A(X)$ times the limiting boundary Gram operator. These identities and limits are proved in `TNLean/MPS/ParentHamiltonian/LocalObservableInsertion.lean`. The required general compression inequality is proved in `TNLean/MPS/ParentHamiltonian/GroundSpaceCompressionNorm.lean`: for an injective boundary map T , with $K = (T^*T)^{-1}$ and range projection P , it gives

$$\|PXP - cP\| \leq \|K\| \|T^*XT - cT^*T\|. \quad (8)$$

An eventual uniform bound on K transfers vanishing of the pullback defect to vanishing of the full compression defect, even when the Hilbert spaces vary. In particular, for $\eta_A(X) = 0$, the complete ground-space projection of $X\psi_n$ tends to zero uniformly over unit boundary ground vectors. This consequence and local expectation convergence are proved in `TNLean/MPS/ParentHamiltonian/BoundaryObservableCompression.lean`. The finite expectations are positive, normalized, complex linear, and consistent under adjoining identity sites. Their operator norm is one; these state properties are proved in `TNLean/MPS/ParentHamiltonian/LocalObservableExpectation.lean` and `TNLean/MPS/ParentHamiltonian/LocalObservableState.lean`.

The fixed local commutator identity is also proved. For an interaction h of range $R \geq 1$ and an observable X on $k \geq 1$ sites, put $t = R - 1$, $w = 2t + k$, and $B = I_t \otimes X \otimes I_t$. If both free intervals have at least t sites, then $X_n^*[H_n(h), X_n]$ is the identity-tensor inclusion of $B^*[H_w(h), B]$. This locality identity is in `TNLean/MPS/ParentHamiltonian/BulkObservableCommutator.lean`. Combining it with the primitive expectation and full projection limits gives

$$\gamma \operatorname{Re} \eta_A(X^*X) \leq \operatorname{Re} \eta_A(B^*[H_w(h), B]). \quad (9)$$

For every positive interaction h with kernel $G_R(A)$, a positive γ independent of X is derived at every $R \geq D^4 + 1$, where $D > 0$ is the bond dimension. Comparison with the canonical interaction preserves the open-chain kernel and gives a uniform gap. There is no supplied energy-limit assumption and no finite-volume uniqueness assumption. The canonical and positive-interaction statements are proved in `TNLean/MPS/ParentHamiltonian/PrimitiveLocalCommutatorGap.lean` and `TNLean/MPS/ParentHamiltonian/PrimitiveLocalParentInteractionGap.lean`.

The consistent local expectations now have a continuous positive extension ω_A to the quasi-local algebra, with $\omega_A(I) = \|\omega_A\| = 1$. Finite-region compatibility, interval evaluation, and translation invariance are proved in `TNLean/MPS/ParentHamiltonian/LocalObservableRegionExpectation.lean`, `TNLean/MPS/ParentHamiltonian/LocalObservableQuasiLocalState.lean`, and `TNLean/MPS/ParentHamiltonian/LocalObservableTranslationInvariance.lean`. If $j_{a,k}$ includes a k -site matrix observable at position a , then $\omega_A(j_{a,k}(X)) = \eta_A(X)$. Consequently (9) is also the quasi-local inequality $\gamma \operatorname{Re} \omega_A(j_{a,k}(X)^* j_{a,k}(X)) \leq \operatorname{Re} \omega_A(j_{a-t,w}(B^*[H_w(h), B]))$. This formulation is proved in `TNLean/MPS/ParentHamiltonian/QuasiLocalPrimitiveGap.lean`.

6 The multiblock projection limit and sector purity

For several sectors, the ground-space projection also contains the other sectors. Their local means need not vanish when $\eta_A(X) = 0$ in the chosen sector. These cross-sector components are now removed without asserting scalarity on the entire multiblock ground space. A ground vector in sector α lies in its prefix support. A distant observable commutes with the prefix projections, so

$$\|P_\beta(N)X_{\ell,r}P_\alpha(N)\| \leq \|P_\beta(\ell)P_\alpha(\ell)\| \|X\| \longrightarrow 0 \quad (\alpha \neq \beta). \quad (10)$$

This estimate is proved in `TNLean/MPS/ParentHamiltonian/BoundaryCrossObservableCompression.lean`. The diagonal compression tends to zero by centering. The sum of the sector projections approaches the projection onto their joint span. Thus all column compressions vanish in norm, and so does the full ground projection of $X\psi_n$ for bounded vectors in the chosen sector. The finite-sector assembly and tensor consequence are proved in `TNLean/MPS/ParentHamiltonian/GroundSpaceSectorCompressionDecay.lean` and `TNLean/MPS/ParentHamiltonian/BlockBoundaryObservableCompression.lean`.

For a weighted direct sum of normalized primitive blocks with nonzero weights, simultaneous injectivity at $S > 0$ gives the exact joint kernel and a uniform gap at every $R \geq S + 1$. Every positive interaction with this local kernel consequently has one positive commutator constant for all sectors and centered local observables. The finite-interval statement and its formulation in the constructed quasi-local sector states are proved in `TNLean/MPS/ParentHamiltonian/MultiblockLocalCommutatorGap.lean` and `TNLean/MPS/ParentHamiltonian/QuasiLocalMultiblockGap.lean`. No energy limit, projection limit, or finite-volume uniqueness is supplied as a hypothesis of these existence results. A further corollary starts with the original simultaneous word-spanning family and constructs primitive representatives with faithful invariant matrices, preserving every block ground space and the original positive interaction.

The exact local support of the completed state is also proved in `TNLean/MPS/ParentHamiltonian/LocalParentExpectation.lean`: an observable annihilating $G_k(A)$ has expectation zero at every position. In particular, $\omega_A(j_{a,N}(P_{G_N(A)})) = 1$ for every interval, including $N = 0$. For any competing normalized positive state φ with the same support constraints, Cauchy–Schwarz gives $\varphi(PXP) = \varphi(X)$ on each interval. Primitive scalar compression on expanding intervals then identifies all its local expectations with those of ω_A . Continuity proves $\varphi = \omega_A$ on the completed algebra. In a convex decomposition of ω_A , positivity forces both constituents to inherit each support constraint. Uniqueness therefore proves purity among all states, without imposing translation invariance on the constituents. These statements are proved in `TNLean/MPS/ParentHamiltonian/PrimitiveQuasiLocalUniqueness.lean` and `TNLean/MPS/ParentHamiltonian/PrimitiveQuasiLocalPurity.lean`.

The literal quasi-local commutator observable is eventually equal to the fixed patch, as proved in `TNLean/MPS/ParentHamiltonian/QuasiLocalCommutatorLocality.lean`. Local support and positivity make its sector expectation real and nonnegative, by `TNLean/MPS/ParentHamiltonian/LocalCommutatorExpectation.lean`. Consequently the literal limit in (1) now exists and satisfies the complex-order inequality in each constructed pure sector, with one common positive constant at $R \geq S + 1$. This is proved in `TNLean/MPS/ParentHamiltonian/QuasiLocalMultiblockLimitGap.lean`. The zero-energy face is now classified for a given finite family of inequivalent primitive sectors under the eventual kernel hypothesis, as described below. The remaining passage from arbitrary periodic GVBS presentations to this family is separate.

7 Support from the eventual kernel condition

Let $h \geq 0$ have positive range R , and suppose that $\ker H_N(h) = G_N$ for all sufficiently large N . A normalized positive state φ satisfying $\varphi(\tau_a(h)) = 0$ for every $a \in \mathbb{Z}$ then satisfies $\varphi(P_{G_N}) = 1$ on every sufficiently large interval. Indeed, positivity and finite dimensionality give $H_N(h) \geq \kappa_N(1 - P_{G_N})$ for some $\kappa_N > 0$. Applying φ shows that $\varphi(1 - P_{G_N}) = 0$. No uniform lower bound on κ_N , translation invariance of φ , or parent-kernel identity at the original range is required. This implication is proved in `TNLean/MPS/ParentHamiltonian/QuasiLocalGroundStateSupport.lean`.

For finitely many inequivalent primitive sectors with faithful invariant matrices, eventual joint support also determines the entire state:

$$\varphi = \sum_{\alpha} t_{\alpha} \eta_{\alpha}, \quad t_{\alpha} \geq 0, \quad \sum_{\alpha} t_{\alpha} = 1. \quad (11)$$

The coefficients are derived from expectations of the sector projections on expanding centered intervals. Asymptotic orthogonality makes their total mass tend to one; compactness supplies limiting coefficients. Compression of each local observable identifies its expectation, and continuity extends the identity to the completed quasi-local algebra. Translation invariance of φ is not assumed. This implication is proved in `TNLean/MPS/ParentHamiltonian/MultiblockQuasiLocalFace.lean`.

Conversely, eventual annihilation of the joint MPS space by the open-chain sums gives zero local expectation in every constructed sector state. Each summand has the same expectation in such a state, and a volume of length at least R contains at least one summand. Positivity is unnecessary for this converse implication, proved in `TNLean/MPS/ParentHamiltonian/LocalParentExpectationFromOpenKernel.lean`.

The zero-energy face therefore equals the convex hull in (11). Since each primitive sector state is pure among all quasi-local states, and a pure state in a finite convex decomposition equals a constituent, the pure zero-energy states are exactly the constructed sectors. Both equivalences are proved in `TNLean/MPS/ParentHamiltonian/MultiblockQuasiLocalGroundStates.lean`. This classification concerns the given finite primitive sector family; more general periodic GVBS presentations are not thereby classified.

8 One constant for finite chains and all pure ground states

The original interaction range need not satisfy $R \geq S + 1$ when the source's eventual finite-volume kernel identity is assumed. Choose a sufficiently large canonical range $W \geq R$ with uniform gap $\delta > 0$ and with both open-chain kernels equal to the joint MPS space. Positivity and finite dimensionality give

$$\kappa P_{G_W^+} \leq H_W(h) \quad \text{for some } \kappa > 0. \quad (12)$$

Sum this comparison over complete W -site windows in an N -site chain. Every original range- R term occurs at most $W - R + 1$ times, so

$$\kappa H_{N,W}^{\text{can}} \leq (W - R + 1) H_N(h). \quad (13)$$

The coincident kernels transfer the canonical gap to the lower bound $\kappa\delta/(W - R + 1)$. A minimum with the positive gaps of the finitely many shorter volumes gives a common constant $\gamma > 0$ for every length. The comparison and its consequence are proved in `TNLean/MPS/ParentHamiltonian/Martingale/OpenInteractionRangeComparison.lean` and `TNLean/MPS/ParentHamiltonian/Martingale/EventualKernelOpenGap.lean`.

Full joint projection decay and finite-range locality then give (1) in every constructed sector with this same γ . The ground-state classification identifies every pure zero-energy state with such a sector, so the bound holds in every pure state of the zero-energy face. This paired finite- and infinite-volume conclusion is proved in `TNLean/MPS/ParentHamiltonian/QuasiLocalGroundStateGap.lean`. Simultaneous word span of an original, unnormalized family suffices: normalization preserves the joint spaces at every length, and the range of h remains arbitrary and positive.

For a given inequivalent primitive family, a positive canonical interaction with these ground states is also constructed in `TNLean/MPS/ParentHamiltonian/PrimitiveGroundStateInteractionExistence.lean`. Its range R is derived, its finite-chain kernel equals the prescribed MPS space for every $N \geq R$, and its zero-energy face and pure states have the stated classifications. These are the existence and gap assertions for the specified primitive representation.

9 Blocking and periodic tensor representations

The support and purity assertions also hold on the original chain when a primitive presentation is obtained only after blocking. Let $L > 0$ and suppose

$$G_{A^{[L]}}(N) = G_{\bigoplus_j \mu_j B_j}(N) \quad (N \geq 0), \quad \mu_j \neq 0, \quad (14)$$

where the B_j are inequivalent primitive tensors with faithful invariant matrices. The blocking isomorphism induces a transport T_L identifying the entire state spaces and their pure states.

For any normalized positive state, support on every original interval is equivalent to support on every interval aligned with the block partition. Every interval lies in an aligned interval. Restriction of the larger MPS space orders the corresponding projections, and exact transport gives

$$\varphi(P_{G_A(I)}) = 1 \quad \text{for every nonempty } I \quad \Longleftrightarrow \quad T_L^{-1}\varphi(P_{G_{A[L]}(J)}) = 1 \quad \text{for every nonempty blocked } J. \quad (15)$$

The primitive classification and (14) imply that all such states are convex combinations of $T_L\omega_j$, and that their pure states are exactly the $T_L\omega_j$. These assertions are proved in `TNLean/MPS/ParentHamiltonian/BlockedQuasiLocalFace.lean`. Competing states need not be invariant under any translation.

For a periodic tensor, the primitive cyclic sectors after period-blocking give the required representation. Isometric compression preserves every finite MPS space, and primitive transfer maps give faithful invariant matrices. Removing gauge-phase repetitions preserves the joint spaces. Thus (14) holds for a nonempty finite family of inequivalent primitive tensors; no sector matrices or inequivalence hypotheses are supplied separately. This construction is proved in `TNLean/MPS/ParentHamiltonian/PeriodicGroundStatePresentation.lean`.

There is also an exact criterion relating local zero energy to interval support. For any tensor A , a positive interaction of positive range whose eventual open-chain kernel is $G_A(N)$ satisfies

$$\varphi \in \mathcal{F}_h \quad \Longleftrightarrow \quad \varphi \text{ is a normalized positive state and } \varphi(P_{G_A(I)}) = 1 \text{ for every nonempty } I. \quad (16)$$

Projection ordering extends eventual symmetric support to all intervals. Conversely, each interaction term annihilates the MPS space of a sufficiently large containing volume. This criterion is proved in `TNLean/MPS/ParentHamiltonian/EventualKernelGroundStateSupport.lean`. Together with the periodic presentation it classifies the entire original zero-energy face and its pure states in `TNLean/MPS/ParentHamiltonian/PeriodicQuasiLocalGroundStates.lean`.

Nonwrapping interaction windows commute exactly with configuration regrouping. To compare Hamiltonians, let $\tilde{h}$ be $H_h(KL)$ regrouped as an interaction on K blocked sites. If $R + L \leq KL + 1$, then for $N \geq K$, in the original configuration coordinates,

$$H_h(NL) \leq H_{\tilde{h}}(N) \leq (KL + 1 - R)H_h(NL). \quad (17)$$

Every original window is covered by an aligned KL -site window; each occurs at most $KL + 1 - R$ times. Positivity therefore gives coincident kernels. These identities and the eventual blocked kernel condition are proved in `TNLean/MPS/ParentHamiltonian/BlockedInteractionKernel.lean`.

Apply the primitive gap theorem to the blocked interaction. Configuration reindexing preserves the entire kernel and its orthogonal complement. Thus a blocked gap γ transfers to the original gap $\gamma/(KL + 1 - R)$ for $N \geq K$. A positive minimum with the finitely many shorter divisible volumes gives one constant for every length NL , using the actual kernel at each volume. For a periodic tensor this gives a uniform gap at every period-divisible volume, with its primitive sector data derived internally. The transfer is proved in `TNLean/MPS/ParentHamiltonian/BlockedFiniteGapTransport.lean`. Section 11 below extends the finite-volume estimate to every residue length for a single normalized periodic tensor.

For a single normalized periodic tensor, Section 10 below constructs the original interaction with these exact open kernels. Identification of the support spaces arising from a general GVBS construction with these tensor spaces remains separate. Accordingly, the unrestricted source Theorems 1.1 and 1.2 remain separate from the periodic-tensor and primitive-family conclusions established here.

10 The original-chain commutator gap for periodic tensors

The infinite-volume inequality is now proved directly on the original chain for a periodic tensor A . Its hypotheses are positive physical dimension, $R > 0$, $h \geq 0$, and the eventual original kernel identity $\ker H_h(N) = G_A(N)$. The primitive sectors, faithful matrices, and their inequivalence are derived from periodicity. There is one $\delta > 0$ such that every pure state $\varphi \in \mathcal{F}_h$ and every local X with $\varphi(X) = 0$ satisfy

$$\lim_{u,v \rightarrow \infty} \varphi(X^*[H_{[a-u, a+k+v]}, X]) \geq \delta \varphi(X^*X). \quad (18)$$

The limit is literal, with complex order as in the source, and both margins may diverge along any filter. Translation invariance of a competing state is not assumed. This is proved in `TNLean/MPS/ParentHamiltonian/PeriodicQuasiLocalGroundStateGap.lean`.

The same conclusion holds for the joint tensor $T = \bigoplus_j \mu_j A_j$ of a finite family of normalized periodic tensors with nonzero weights. Blocking by the product of their periods gives an exact primitive presentation at every length, with faithful invariant matrices and inequivalence derived internally. The constant is chosen before all states, observables, filters, and margins. This extension is proved in `TNLean/MPS/ParentHamiltonian/PeriodicBlockFamilyQuasiLocalGap.lean`. An empty family has an empty zero-energy state face; no ground-state existence is asserted in that case.

The comparison uses positivity before taking a limit. If a positive functional f satisfies $f(H) = 0$ for $H \geq 0$, Cauchy-Schwarz implies $f(YH) = 0$ for every Y . Thus

$$f(X^*[H, X]) = f(X^*HX). \quad (19)$$

If $0 \leq C \leq MH$, then $f(C) = 0$ and

$$f(X^*[C, X]) \leq Mf(X^*[H, X]). \quad (20)$$

Apply this to the blocked Hamiltonian comparison, where $M = KL + 1 - R$. Finite-range locality identifies each commutator energy with one fixed patch once both margins contain the interaction range. These identities transport a blocked gap γ to the original constant γ/M .

Every original local observable can be extended by identities to an aligned interval and then written in blocked coordinates. The stabilized commutator energy depends only on the quasi-local observable, not on the chosen interval representation. Hence the estimate applies also to observables crossing block boundaries and to negative interval starts. Its reality and nonnegativity follow from the positive-functional identity above; no energy convergence is supplied as a hypothesis. The generic transport is proved in `TNLean/MPS/ParentHamiltonian/BlockedQuasiLocalGap.lean`. Section 11 gives the finite-volume estimate at every original length.

11 Construction on the original periodic chain

For a normalized tensor of period m , the original interaction and its kernel identity are now derived rather than assumed. Let $(P_u)_{u \in \mathbb{Z}/m\mathbb{Z}}$ be the Hermitian orthogonal cyclic projections, with $\sum_u P_u = I$ and $P_u A_i = A_i P_{u+1}$. Define

$$W_n = \{X : P_u X = X P_{u+n} \text{ for every } u\}, \quad (21)$$

$$V_n = \{X : P_{u+n} X = X P_u \text{ for every } u\}. \quad (22)$$

Every word of length n belongs to W_n . Simultaneous word spanning of the inequivalent primitive tensors obtained by period-blocking yields a length qm whose words span W_{qm} . All sector dimensions, invariant matrices, and inequivalence assertions are derived from the periodic tensor.

The spanning equality propagates to every later original length. If $X \in W_{qm+r}$ and A_w is a word of length r , then $XA_w^* \in W_{qm}$. Left-canonical normalization gives

$$X = \sum_{|w|=r} (XA_w^*) A_w, \quad \sum_{|w|=r} A_w^* A_w = I. \quad (23)$$

Thus the words of length $qm+r$ span W_{qm+r} . Every local MPS vector has a boundary representative in V_n : replace an arbitrary boundary X by $\sum_u P_{u+n} X P_u$ without changing its trace pairing with the length- n words. Moreover, $X \in V_n$ implies $X^* \in W_n$. When W_n is spanned, vanishing of the boundary vector therefore implies $\text{tr}(X^* X) = 0$, hence $X = 0$. The boundary map is injective on the relevant cyclic space, without injectivity on the full matrix algebra.

Comparing the two one-site restrictions and using this restricted injectivity gives, for every sufficiently large n ,

$$G_A(n+2) = (G_A(n+1) \otimes \mathbb{C}^d) \cap (\mathbb{C}^d \otimes G_A(n+1)). \quad (24)$$

Choose R above this threshold and let h be the orthogonal projection onto $G_A(R)^\perp$. Positivity of the open sum makes its kernel the simultaneous kernel of all local terms. Iterating the intersection then gives

$$h \geq 0, \quad \ker H_h(N) = G_A(N) \quad (N \geq R). \quad (25)$$

This construction includes every original length. It is proved in `TNLean/MPS/ParentHamiltonian/PeriodicOriginalIntersection.lean`.

Section 9 now applies to this derived interaction. Consequently periodicity alone gives one positive δ , the exact open kernels above, the zero-energy face classification of Section 8, and the literal infinite-volume inequality for every pure zero-energy state and every centered local observable. The constant precedes all states and exhaustions. The existence and gap composition is proved in `TNLean/MPS/ParentHamiltonian/PeriodicParentInteractionGap.lean`. The finite-volume gap at every original length is established in Section 11. Identification of a general GVBS presentation with these tensor support spaces remains separate.

12 The finite-chain gap at every original length

Let A be the normalized periodic tensor of Section 10, with period m . The canonical interaction there admits a constant $\delta > 0$ such that

$$\delta \|v\| \leq \|H_h(N)v\| \quad (N \geq 0, \quad v \perp \ker H_h(N)). \quad (26)$$

For $N \geq R$, its kernel is $G_A(N)$, and positivity gives the equivalent spectral lower bound

$$\langle v, H_h(N)v \rangle \geq \delta \|v\|^2 \quad (N \geq R, \quad v \perp G_A(N)). \quad (27)$$

In particular, the lower bound is independent of the residue of N modulo m . This is a positive lower bound; it does not identify the explicit numerical constant in the finite-gap proposition of [Nac96], lines 2593–2614 of the local source.

Write $N = (K + M)m + r$, where $0 \leq r < m$. Cover the interval by its initial $(K + M)m$ sites and its terminal $Mm + r$ sites. Their common interval has Mm sites. In the blocked cyclic-sector coordinates, denote the right, left, and full boundary maps of sector j by T_j, L_j, C_j . Their joint ranges are precisely the transported original ground spaces of these three intervals.

For a single sector, the supported Gram inverses give the exact identity

$$P_{\text{ran } T_j} P_{\text{ran } L_j} - P_{\text{ran } C_j} = T_j H_{T_j} \mathcal{D}_j H_{L_j} L_j^*. \quad (28)$$

The finite mixed Gram displacement $\mathcal{D}_j$ tends to zero as $M \rightarrow \infty$. For all sufficiently large M , the two boundary maps and two supported inverses have bounds uniform in K and r . Consequently the same-sector projection defect tends to zero. Products involving two distinct sectors also tend to zero, by the primitive prefix-angle estimate, itself derived from mixed-transfer decay. Finally, each joint range projection is approximated by the sum of the individual sector projections. Combining these three assertions gives

$$\|P_{G_A(N)} - P_{\text{left}} P_{\text{right}}\| \rightarrow 0 \quad (M \rightarrow \infty) \quad (29)$$

along every choice of K and r . A surjective configuration isometry preserves this norm; taking the adjoint puts the two projections in the displayed order. The estimates and their physical transport are proved in `TNLean/MPS/ParentHamiltonian/ResidualWindowJointProjectionDecay.lean` and `TNLean/MPS/ParentHamiltonian/ResidualCoverProjectionDefect.lean`.

The convergence is uniform in the exterior lengths. Indeed, if there were arbitrarily large M for which some pair (K, r) had defect greater than $1/2$, choosing such a pair at each of those M would contradict the convergence along every choice. Fix M sufficiently large that the defect is at most $1/2$ for all exterior lengths and that $Mm \geq R$.

The left interval has the aligned gap of Section 8. The finitely many terminal volumes $Mm + r$, $0 \leq r < m$, have a positive minimum gap above their actual kernels. The exact original-chain intersection identifies the intersection of the two covered kernels with $G_A(N)$. Every local term occurs in the two covered Hamiltonians at most twice, and hence

$$H_{\text{left}} + H_{\text{right}} \leq 2H_h(N). \quad (30)$$

The two-projection gap estimate now gives one positive lower bound for all $N \geq Mm$. The finitely many shorter volumes each have a positive lower bound on the orthogonal complement of their actual kernel. Taking one further finite minimum gives the displayed estimate at every N . The construction involves neither a divisibility condition on N nor additional sector, kernel, or decay hypotheses. The common interaction range, exact kernels for $N \geq R$, and the all-length gap are proved together in `TNLean/MPS/ParentHamiltonian/PeriodicOpenParentGap.lean`. The identification of the support spaces of a general GVBS presentation with these tensor boundary spaces remains separate.

The finite- and infinite-volume conclusions hold for one and the same canonical interaction. More precisely, periodicity chooses a positive range R , a positive matrix h , and constants $\delta, \gamma > 0$ such that the finite-chain estimate above holds with δ , while every pure zero-energy state satisfies the literal infinite-volume commutator inequality of Section 9 with γ . The two constants need not agree. All four choices precede the state, local observable, and interval exhaustion. This combined statement is proved in `TNLean/MPS/ParentHamiltonian/PeriodicParentGap.lean`.

13 Stationary faithful generators and their parent gaps

If, in addition, $E_A(\rho) = \rho$, every invariant projection is reducing. The finite-dimensional reducing decomposition gives a nonempty family of irreducible corners B_j and isometries V_j satisfying

$$V_j^* V_j = I, \quad \sum_j V_j V_j^* = I, \quad A_i V_j = V_j B_j^i, \quad V_j^* A_i = B_j^i V_j^*.$$

Each corner remains trace preserving, since $\sum_i (B_j^i)^* B_j^i = V_j^* (\sum_i A_i^* A_i) V_j = I$. Irreducibility and this normalization give a finite period for each corner. The isometric resolution identifies the support spaces at

every length:

$$G_A(N) = \bigvee_j G_{B_j}(N) = G_{\bigoplus_j B_j}(N) \quad (N \geq 0).$$

The full presentation, including both intertwining directions, is proved in `TNLean/MPS/ParentHamiltonian/FaithfulPeriodicGroundSpace.lean`. No minimality or purity assumption is used, and repeated corners are allowed.

For stationary data, the compatible finite expectations define a quasi-local state $\omega_{A,\rho}$. The same density σ_k represents every translated interval restriction of this state, and its range remains $G_A(k)$. This literal state-support statement is proved in `TNLean/MPS/ParentHamiltonian/FaithfulLocalStateSupport.lean`.

The periodic presentation also gives an infinite-volume gap for these supplied generators without assuming that A is irreducible or periodic. Let $h \geq 0$ have positive range and suppose that $\ker H_h(N) = G_A(N)$ for every sufficiently large N . There is a single $\gamma > 0$ such that, for every pure zero-energy state φ , every centered local observable X , and every exhaustion with both boundary margins diverging,

$$\lim_{\Lambda \nearrow \mathbb{Z}} \varphi(X^*[H_{h,\Lambda}, X]) \geq \gamma \varphi(X^*X).$$

The periodic-family gap theorem applies to the same interaction after the all-length support identity. The constant precedes the state, observable, and exhaustion. This conditional exact-kernel consequence is proved in `TNLean/MPS/ParentHamiltonian/FaithfulQuasiLocalGap.lean`.

The interaction and kernel identity can now also be constructed from the faithful stationary generators. Their periodic presentation and the family intersection theorem give $R_0 > 0$ such that

$$\ker H_{R,N}^{\text{open}}(A) = G_A(N) \quad (N \geq R \geq R_0).$$

Choose the canonical projection of range R_0 . It is positive, has these exact open kernels, and annihilates the generated state on every translate. The preceding commutator theorem applies to this same interaction. Thus neither an interaction nor an eventual kernel identity is supplied in `TNLean/MPS/ParentHamiltonian/FaithfulParentInteractionGap.lean`.

The original-length family gap also gives a uniform finite-volume bound for every sufficiently large canonical range. For each such range R there is $\delta_R > 0$ such that, at every $N \geq 0$,

$$\delta_R \|v\| \leq \|H_{R,N}^{\text{open}}(A)v\| \quad \text{for } v \perp \ker H_{R,N}^{\text{open}}(A). \quad (31)$$

The exact boundary-kernel assertion still concerns $N \geq R$. Thus the constructed interaction has both finite and infinite gaps, and contains the generated stationary state in its zero-energy face. This is proved in `TNLean/MPS/ParentHamiltonian/FaithfulParentFiniteGap.lean`.

A given positive interaction h of range $R > 0$ needs only the eventual identity $\ker H_h(N) = G_A(N)$. Choose a long window W beyond both this threshold and the canonical threshold. Finite-dimensional positivity gives $H_h(W) \geq \kappa P_{G_A(W)^\perp}$ for some $\kappa > 0$. The sum of the translated windows counts each interaction term at most $W - R + 1$ times. Comparison with the range- W canonical Hamiltonian therefore transfers its gap to the given interaction. A positive minimum over the remaining finitely many volumes extends the bound above their actual kernels. The same given h has the literal infinite-volume commutator gap. No assumption on $\ker h$ at the interaction range is added. These results are proved in `TNLean/MPS/ParentHamiltonian/Martingale/CanonicalGapToInteractionGap.lean` and `TNLean/MPS/ParentHamiltonian/FaithfulGivenInteractionGap.lean`. They give Theorem 1.2 in the supplied faithful stationary-generator setting. For a positive finite-volume Hamiltonian, the norm gap is equivalent to the lower bound on its nonzero spectrum; it therefore gives $\langle v, H_h(N)v \rangle \geq \delta \|v\|^2$ above the kernel. Taking the positive minimum of the finite and infinite constants gives one common lower bound for both conclusions, as in the source statement.

Faithfulness can be obtained by restricting supplied stationary generating data to their support. Suppose only that $\rho \geq 0$ has nonzero trace and $E_A(\rho) = \rho$. Its support projection P is invariant: $(I - P)A_iP = 0$. Choose an isometry V with $VV^* = P$, and put

$$B_i = V^* A_i V, \quad \sigma = V^* \rho V.$$

Then $\sigma > 0$, $\sum_i B_i^* B_i = I$, $E_B(\sigma) = \sigma$, and $A_i V = V B_i$. Moreover,

$$\rho = V \sigma V^*, \quad \text{tr } \sigma = \text{tr } \rho, \quad \eta_{A,\rho}(X) = \eta_{B,\sigma}(X)$$

at every finite length, including zero. These assertions are proved in `TNLean/MPS/ParentHamiltonian/StationarySupportCompression.lean`. The compressed bond dimension is positive because its stationary trace is nonzero. This restriction preserves the specified state; it need not preserve the original tensor's full boundary spaces. Invariance of P alone does not give $V^* A_i = B_i V^*$, and unused original directions can enlarge $G_A(k)$. For every $k \geq 0$, the original quasi-local state nevertheless has one positive density of trace one on every translated k -site interval, whose range is exactly $G_B(k)$. This follows from the preserved finite expectations and the faithful support theorem for B, σ , and is proved in `TNLean/MPS/ParentHamiltonian/StationaryLocalStateSupport.lean`.

Equality of the finite insertion expectations also identifies the actual quasi-local states:

$$\omega_{A,\rho} = \omega_{B,\sigma}.$$

Every finite-region observable can be placed in a containing interval, where the two functionals agree. Continuity and density of the local observables then give equality on the quasi-local algebra. This is proved in `TNLean/MPS/ParentHamiltonian/QuasiLocalExpectationEquality.lean`. Thus the support restriction preserves the entire specified state, rather than merely its finite density ranges.

The compressed faithful generators therefore construct a positive parent interaction for the original stationary state. Its sufficiently long open kernels are $G_B(N)$, the original state belongs to its zero-energy face, and the same interaction has one positive infinite-volume commutator gap in every pure state in that face. This construction assumes only normalized generators and a nonzero positive stationary matrix; bond and physical dimensions are derived. It is proved in `TNLean/MPS/ParentHamiltonian/StationaryParentInteractionGap.lean`. The original stationary state need not be pure. The same construction now also supplies $\delta > 0$ with

$$\delta \|v\| \leq \|H_h(N)v\| \quad \text{for every } N \geq 0 \text{ and } v \perp \ker H_h(N). \quad (32)$$

The constants for the finite and infinite gaps belong to this same interaction. The all-length result is proved in `TNLean/MPS/ParentHamiltonian/StationaryParentFiniteGap.lean`.

For a supplied physical-first isometry $V : \mathbb{C}^D \rightarrow \mathbb{C}^d \otimes \mathbb{C}^D$, put $A_i(a, b) = V((i, a), b)$. Direct matrix multiplication gives

$$\sum_i A_i^* A_i = V^* V, \quad V^*(X \otimes B)V = \sum_{i,j} X_{ij} A_i^* B A_j. \quad (33)$$

Trace duality identifies this one-site Heisenberg insertion with the corresponding Schrödinger insertion. These identities are proved in `TNLean/MPS/ParentHamiltonian/SuppliedIsometryInsertion.lean`. The multi-site order is now explicit as well. Write $F_X(B) = V^*(X \otimes B)V$. For every finite list and every matrix ρ ,

$$\text{Tr}(\rho(F_{X_1} \circ \cdots \circ F_{X_n})(I)) = \text{Tr}(E_{X_n \otimes \cdots \otimes X_1}(\rho)). \quad (34)$$

Here the physical observable transfer on the right uses forward matrix words. The identity includes the empty list and needs no positivity, normalization, or stationarity. Its induction uses one-site trace duality and the product rule for inserted transfers, and is proved in `TNLean/MPS/ParentHamiltonian/SourceOrderedInsertion.lean`. Thus the source-written product has precisely the reversed physical order in the present convention.

These results identify finite supports and periodic sectors once stationary generating data are supplied. They do not construct those data from an arbitrary GVBS presentation. The source discussion of minimum auxiliary dimension is not the assertion that the observable-generated algebra is the whole matrix algebra. The preceding multi-site identity records the required physical reversal for a supplied isometric representation.

14 Original-length periodic overlaps and joint intersections

For a normalized irreducible periodic tensor A , there are constants $C > 0$ and n_0 such that

$$x \in G_A(n), n \geq n_0 \implies \exists u : \Gamma_n^A(u) = x, \|u\| \leq C\|x\|.$$

This bound holds at every original length. Primitive cyclic sectors give supported residual Gram inverses uniformly in the residue length. If S_j is one residual boundary map and H_j its supported Gram inverse, then $S_j H_j S_j^* = P_{\text{ran } S_j}$. The corresponding original boundary is obtained by right multiplication by V_j^* . Small pairwise sector overlaps give

$$\frac{1}{2} \sum_j \|x_j\|^2 \leq \left\| \sum_j x_j \right\|^2.$$

This lower bound controls each component by $2\|x\|$, so summing the individual bounded preimages gives the assertion. Quotient and remainder then include every sufficiently large original length.

For inequivalent periodic tensors A, B , the original mixed transfer operator has spectral radius strictly below one. Hence $\|\Gamma_n^A \Gamma_n^B\| \rightarrow 0$. Combining this decay with the bounded preimages gives, for each $\varepsilon > 0$, eventually

$$|\langle x, y \rangle| \leq \varepsilon \|x\| \|y\| \quad (x \in G_A(n), y \in G_B(n)).$$

The two tensors may have different periods and bond dimensions. The argument uses the original mixed transfer operator and therefore supplies the estimate at every sufficiently large original length. These results are proved in `TNLean/MPS/ParentHamiltonian/PeriodicGroundSpaceIndependence.lean`.

For a finite family, sufficiently small pairwise overlaps imply independence by the lower Gram estimate. On the common middle interval, independence makes the two sector decompositions unique. The individual one-step intersections can consequently be combined:

$$\left(\sum_j G_j(n+1) \otimes \mathbb{C}^d \right) \cap \left(\mathbb{C}^d \otimes \sum_j G_j(n+1) \right) = \sum_j G_j(n+2).$$

This geometric implication is proved without normalization or virtual injectivity in `TNLean/MPS/ParentHamiltonian/JointGroundSpaceIntersection.lean`. Iteration gives the exact canonical open kernels once the individual intersections and middle independence hold eventually; the latter two facts remain explicit in `TNLean/MPS/ParentHamiltonian/PeriodicFamilyInteractionCriterion.lean`.

An arbitrary finite periodic family need not be inequivalent. Select one representative of each gauge and phase class first. Gauge invariance of boundary spaces and nonzero block weights preserve the original joint space at every length. The selected spaces are eventually independent, so the intersection criterion constructs a positive canonical interaction with exact original kernels for every volume at least its range. The periodic-family commutator theorem then supplies a common positive infinite-volume gap for that constructed interaction. No sector separation, independence, interaction, or kernel identity is supplied. The result is proved in `TNLean/MPS/ParentHamiltonian/PeriodicBlockFamilyParentInteractionGap.lean`. It concerns an explicitly supplied finite periodic generating family; the identification with an arbitrary source GVBS boundary-limit presentation remains separate.

The common blocking can also be resolved into literal primitive cyclic sectors. Their isometric inclusions satisfy $\sum_s V_s V_s^* = I$; the residual word Gram has the inverse-cyclic tail projection in each sector. The supported inverse estimates and vanishing mixed overlaps then give the physical residual-cover limit at every original residue. Consequently, for some $R_0 > 0$, every $R \geq R_0$ has exact kernels at $N \geq R$ and a uniform norm gap above the actual kernel at every $N \geq 0$. The same constructed canonical interaction has the literal infinite-volume gap. No primitive-resolution or residual-limit hypothesis is supplied in `TNLean/MPS/ParentHamiltonian/PeriodicBlockFamilyOpenParentGap.lean`. The empty family is included by its direct all-length Hamiltonian formula.

For the original weighted periodic family, a prescribed positive interaction of positive range with eventual exact joint kernels has the same two gap conclusions. The finite-volume bound uses its actual kernel also at short lengths; the infinite-volume bound concerns the identical supplied interaction. No interaction-range lower threshold or local-kernel equality is required. This given-interaction form is proved in `TNLean/MPS/ParentHamiltonian/PeriodicBlockFamilyInteractionGap.lean`.

References

- [Nac96] Bruno Nachtergaele. The spectral gap for some spin chains with discrete symmetry breaking. *Communications in Mathematical Physics*, 175(3):565–606, 1996.