

Exact Preparation at a Zero Subleading Transfer Spectrum

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Verdict: local-correction (resolved).

1 The source prescription

For a normal tensor in trace-preserving gauge, the source writes $\mathcal{E} = P + R$, where $P(X) = \text{tr}(X)\sigma$, $\sigma > 0$, $\text{tr } \sigma = 1$, and R has spectral radius less than one [MSWC24, paragraph “Preliminaries”]. Its logarithmic block prescription uses the correlation length $\xi = -1/\log |\lambda_2|$ and chooses $q = \lceil 2\xi(1 + \eta) \log N \rceil$ after Lemma 1. When all subleading eigenvalues vanish, $\log |\lambda_2|$ is not finite. The prescription must therefore be replaced by a positive finite block length at this endpoint. In particular, a value assigned to the symbol ξ alone is not the spectral hypothesis used below.

2 Nilpotent transients and a finite block length

Assume explicitly that every eigenvalue of $\mathcal{E}$ distinct from 1 is zero. Normality makes the leading fixed projection simple, and trace preservation and stationarity give $P^2 = P$ and $PR = RP = 0$. Every eigenvalue of R is zero. On the D^2 -dimensional matrix space, the generalized eigenspace decomposition gives $R^{D^2} = 0$, including nondiagonalizable maps. Consequently, for every $n \geq D^2$,

$$\mathcal{E}^n(X) = \text{tr}(X)\sigma. \tag{1}$$

The subleading part need not vanish before that finite blocking length.

For a block B of at least D^2 sites, the transfer/polar relation of the source gives $\widehat{B}^\dagger \widehat{B} = \sigma^\top \otimes I$. Thus its positive polar factor is exactly $(\sqrt{\sigma})^\top \otimes I$, and it is injective because $\sigma > 0$. Replacing the polar factors by the fixed-point factors in the source construction therefore changes no state.

Choose $q_0 = D^2 + 3D + 1$. For every $N \geq q_0$, write $N = Mq_0 + r$ with $M \geq 1$ and $0 \leq r < q_0$. Use $M - 1$ blocks of length q_0 and one of length $q_0 + r$. All lengths lie between q_0 and $2q_0$, so the local-isometry preparation applies in depth at most $2C(d, D)q_0$. The raw periodic vector already has norm one, and

the circuit prepares its normalized vector exactly. This bound is independent of N and requires no divisibility condition.

3 Formal scope

The construction is the exact-factor endpoint of the source equations `eq:B_TM`, `eq:phi_tilde`, and `eq:app_tidle_phi_1`; it is not an evaluation of the printed logarithmic prescription at zero correlation length. Its spectral assumption allows nilpotent Jordan blocks and supplies no separate injectivity or diagonalizability certificate. The preparation theorem is `MPSPreparation.exists_isPreparedInDepth_normalizedMPVState_of_subleading_eigenvalues_eq_zero`. It fixes d, D before choosing the depth and applies to every normal left-canonical tensor and normalized positive definite fixed state satisfying the stated zero-subleading-spectrum condition.

This endpoint consequence does not settle the polynomial-accuracy construction at positive correlation length or its arbitrary-length extensions. Those results have their own hypotheses and proofs.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.