

The Tree Circuit as a Multiscale Network: Two-Site Injectivity and Chain Length

2026-09-30; chain-length restriction resolved 2026-10-04

Verdict: scope-restriction (resolved).

1 What the source states

Let A be a normal tensor of physical dimension d and bond dimension D , and let $|\phi_N\rangle$ be the normalized translation-invariant state it generates on a ring of N sites. The paragraph “Connection to MERA” of [MSWC24] reads the tree circuit of eq. (16) as a finite-range multiscale entanglement renormalization ansatz with $O(\log \log N)$ layers: the isometries $V^{(i)}$ of eq. (16) are the isometries of the network, every disentangler is the identity except those of the first layer, which prepare the fixed-point state $|\Omega\rangle$, and hence, within the approximation error ε , normal translation-invariant matrix product states are such networks with $O(\log \log N)$ layers.

The paragraph “The tree-RG circuit” of the source describes the layers as isometries from dimension D^2 to D^4 , except the lowest, and its picture starts from a blocked tensor. A layer $V^{(i)}$ is the partial isometry of the polar decomposition of a two-site blocked map; it is an isometry exactly when that map is injective. For $d^2 < D^2$ the lowest layer, a map from $\mathbb{C}^{D^2}$ to $(\mathbb{C}^d)^{\otimes 2}$, cannot be an isometry at all.

2 The restrictions formerly adopted

The first formal statements¹ assume two things.

1. The two-site blocked tensor of A is injective. Then every layer $V^{(i)}$ is an isometry: blocking the two-site blocked tensor $V^{(1)}T_1$ twice gives $(V^{(1)})^{\otimes 2}$ applied to T_1 blocked twice, so the two-site blocked tensor of T_1 is injective, and so on up the tree.

¹Formal declarations: `MPSPreparation.approximatingMPVState_eq_state_treeMERA`, `MPSPreparation.exists_state_treeMERA_approximationError_le`, and `MPSPreparation.exists_state_treeMERA_approximationError_le_and_le_logb` in `TNLean/MPS/Preparation/TreeMERA`.

2. The chain length is $N = M2^{k+1}$, so that the M blocks of the approximating state of eq. (10) all have $q = 2^{k+1}$ sites and the tree of each block is complete.

Under these assumptions the approximating state of eq. (10) is exactly the state of the network with $k + 1$ isometry layers, and its error against $|\phi_N\rangle$ is at most ε once $2^{k+1} \geq \xi \log(CN/\varepsilon)$, with $\xi = -1/\log t$ for a bound $0 < t < 1$ on the moduli of the subleading eigenvalues of the transfer map and C depending on A and t . For every number M of blocks such a k exists, and the least one satisfies $k \leq \log_2 \max(1, \xi \log(CN/\varepsilon))$. These statements remain, as the case of injective two-site blocked tensors.

3 Resolution of the injectivity restriction

The finest isometries of the network now write *registers* of s sites, where A blocked over s sites is injective, as in the source's picture, which starts from a blocked tensor.² Let B be A blocked over s sites. The two-site blocked tensor of B is A blocked over $2s$ sites, which is injective, so the network of B is defined, with finest isometry $\mathbb{C}^{D^2} \rightarrow \mathbb{C}^{d^s} \otimes \mathbb{C}^{d^s}$. Blocking B over 2^{k+1} sites is blocking A over $q = s2^{k+1}$ sites, read through the grouping of the sites into registers, so the state of the network of B , read on the $N = Ms2^{k+1}$ sites, is the approximating state of eq. (10) of A itself with block length q . The approximation error of A applies directly, so no gauge data need to be transported to B : there is $C > 0$, independent of s , such that the error is at most ε once $s2^{k+1} \geq \xi \log(CN/\varepsilon)$, and the least such k satisfies $k \leq \log_2 \max(1, \xi \log(CN/\varepsilon))$.

For an arbitrary normal tensor A with $D \geq 1$, the gauge of eq. (5) is reached by a gauge transformation and a rescaling, which changes the normalized state by a phase only, and the blocked tensors of the gauged tensor are injective from some length L on. Hence there are $s_0 \geq 1$, $\xi > 0$ and $C > 0$ such that for every $s \geq s_0$, every $\varepsilon > 0$ and every $M \geq 1$ some network with sites of dimension d^s and $k + 1$ isometry layers, $k \leq \log_2 \max(1, \xi \log(CN/\varepsilon))$, has error at most ε against $|\phi_N\rangle$, $N = Ms2^{k+1}$. Every normal tensor is covered.

4 The former chain-length restriction

The chain length is $N = Ms2^{k+1}$ with k given by the threshold: the network consists of M complete binary trees on equal registers. The layer count holds along the chain lengths so obtained, not for every fixed N and ε with registers of bounded length: for $N = 2p$ with p odd only $k = 0$ is available, so the threshold forces $s \geq \xi \log(CN/\varepsilon)/2$, and the dimension d^s of the sites of the network grows with N/ε .

²Formal declarations: `MPSPreparation.BinaryMERA.registerState`, `MPSPreparation.approximatingMPVState_eq_registerState_treeMERA`, `MPSPreparation.exists_registerState_treeMERA_approximationError_le`, `MPSPreparation.exists_registerState_treeMERA_approximationError_le_and_le_logb`, and `MPSPreparation.exists_binaryMERA_registerState_approximationError_le` in `TNLean/MPS/Preparation/TreeMERARegisters`.

5 Resolution for every positive chain length

A sibling construction uses unequal positive blocks and distinct isometries at each vertex. Its tensors are actual binary trees: a leaf is an isometry from $\mathbb{C}^{D'^2}$ to a physical interval of at most R sites, and a binary vertex is an isometry from $\mathbb{C}^{D'^2}$ to two copies of that space. A top unitary prepares the pair joining neighboring roots; all lower disentangles are identities. Thus the state is an explicit tensor contraction, with internal dimensions at most D^2 and physical leaf dimensions at most d^R .³

For every normal A there are constants $R \geq 1$ and $C \geq 2$, chosen before N and ε , such that, for every $\varepsilon > 0$ and every positive N with nonzero unnormalized periodic vector, there is a unit network with some $1 \leq D' \leq D$, error at most ε , and total layer count bounded by

$$h + 2 \leq C(1 + \log(1 + \max\{0, \log(N/\varepsilon)\})).$$

The regularization makes the bound meaningful also for $N = 1$ or $\varepsilon \geq 1$; it is $O(\log \log(N/\varepsilon))$ in the large-size, small-error regime. For $D \geq 1$, normality gives nonvanishing for all sufficiently large N , yielding an eventual statement without a separate nonzero-state hypothesis.⁴ The nonzero-state qualification is intrinsic to normalization, as recorded in the note on the nonzero-state condition.

Choose a left-canonical representative and an injectivity length s , and set $R = 4s$. Repeated floor/ceiling halving constructs a polar tree of coarse depth h on every block of length n satisfying $2^h s \leq n \leq 2^h R$. The polar-merge identity proves that its contraction is exactly the polar isometry of that block.⁵ For the accuracy-selected logarithmic scale $q \leq N$, take all but the last block of length q , with the last of length $q + N \bmod q < 2q$. One common h gives the fixed leaf bound R for every tree. The contraction is the existing unequal-block approximating state, so its second-order error estimate applies. That estimate has rate $\exp(-2\gamma q/\xi)$ for every fixed $0 < \gamma < 1$; the proof uses $\gamma = 1/2$. For $s \leq N < q$, use one polar tree over N and the normalized trace pair at its root. Its contraction is exactly the target normalized state, while its physical leaves still have at most R sites. The root pair may depend on N ; it is not required to be the fixed-point pair in this exact branch. For $N < s$, a single bounded leaf with $D' = 1$ represents the state exactly. This finite set of short lengths does not enlarge R with either N or ε . For $\varepsilon > 1$, a normalized product-state network suffices. These branches remove the restriction for every positive admissible length, including lengths below the accuracy-selected scale.

³Formal declarations: `MPSPreparation.IsometryTree`, `MPSPreparation.IsometryTree.matrix`, and `MPSPreparation.IsometryTree.isIsometry_matrix` in `TNLean/MPS/Preparation/IsometryTree`; `MPSPreparation.UnequalMERA`, `MPSPreparation.UnequalMERA.state`, and `MPSPreparation.UnequalMERA.norm_state` in `TNLean/MPS/Preparation/UnequalMERA`.

⁴Formal declarations: `MPSPreparation.exists_unequalMERA_every_length` and `MPSPreparation.exists_unequalMERA_of_normal` in `TNLean/MPS/Preparation/UnequalTreeMERA`.

⁵Formal declaration: `MPSPreparation.exists_isometryTree_matrix_eq_cfgPolarIso` in `TNLean/MPS/Preparation/PolarIsometryTree`.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.