

The Tree Circuit with Measurements: Rounds, Registers, and Chain Length

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Verdict: scope-restriction (open).

1 What the source states

The paragraph “Tree-RG circuit with measurements” of [MSWC24] observes that the isometries of the tree circuit, eq. (16), “act on a constant number of sites which, although spatially separated, can be teleported at neighboring registers with a constant overhead”, by “creating nearest-neighbor entangled pairs, then performing simultaneous measurements, and correcting (without postselection) based on the measurement outcomes”. Hence “every isometry in eq. (16) takes constant time using measurement”, the tree circuit has $O(\log \log(N/\varepsilon))$ layers, and, with the fixed-point state prepared in constant depth, a short-range correlated matrix product state is prepared with measurements in depth $O(\log \log(N/\varepsilon))$. The source fixes no model of circuits with measurements.

2 What is formalized

1. *The model.* A measurement round is a local circuit, a measurement of some sites in the computational basis, and a product of single-site unitaries chosen from the outcomes; a preparation is a sequence of rounds, the depth being the total number of layers of their circuits. Every round is a protocol of QCcc_ℓ of [PSC21] of no larger depth; that paper performs a single round and remarks that “one could also define a more general scheme with multiple rounds of LOCC”. A sequence of rounds is a composition of such transformations. The number of rounds is not bounded: the tree with k levels uses $2k$ rounds, whereas in the class $\text{QCcc}_\ell^{(k)}$ of the paragraph “Phases of matter” of [PSC21] the number k of composed transformations does not depend on the system size. Several rounds are needed because a teleported register must be corrected before the next isometry acts on it.
2. *Teleportation.* For qudits of any dimension d , any number of registers is moved along chains of nearest-neighbour hops of any length, in one round

of depth 2; for every outcome the round acts as d^{-H} times the permutation of sites, H the number of hops.

3. *One layer.* Two-site gates on pairwise disjoint stretches $a, \dots, a + 2L + 1$, with the sites strictly inside each stretch in $|0\rangle$, are applied in two rounds of total depth 5, whatever L . A gate joins the sites a and $a + 2L + 1$, at odd separation along the ring; gates at even separation, or on overlapping stretches, are not covered, and the tree needs neither.
4. *The tree.* A binary tree of two-site gates with k levels, on the complete blocks of 2^k sites and on the complete smaller blocks after them, is applied with measurements in depth $5k$, on the vectors with $|0\rangle$ strictly inside the complete blocks of every level.
5. *Registers of several sites.* A register is s consecutive sites. A unitary on two registers $a, \dots, a + s - 1$ and $a + 2L + s, \dots, a + 2L + 2s - 1$ is applied by teleporting the first register site by site, the site $a + s - 1$ first, in s rounds of depth 2, applying the unitary to $2s$ consecutive sites by a local circuit of depth K , and teleporting back; a layer of such unitaries on disjoint stretches takes depth $4s + K + 2$, whatever L . Trees of such unitaries on the M blocks of $q = s2^{k+1}$ sites take depth $(k + 1)(4s + K + 2)$.
6. *The approximating state.* If the blocked tensor of A over s sites is injective, a register carries $\mathbb{C}^{D^2}$, every layer of the tree factorization of the blocked tensor over s sites is an isometry, and the tree of unitaries extending these isometries implements the isometry V of the polar decomposition of A blocked over q sites. After the pairs of the fixed point, prepared by a local circuit, the protocol outputs the approximating state of eq. (10) with blocks of q sites, in depth at most $C(k + 1)$ with C depending on d and s .
7. *The depth.* For a normal tensor, s is a length at which the blocked tensor in the gauge of eq. (5) is injective, and with Lemma 1'(i) at slope $a = \xi$ the error is at most ε once $s2^{k+1} \geq a \log(N/\varepsilon) + b$. If moreover $s2^{k+1} \leq 2(a \log(N/\varepsilon) + b)$, then $k \leq \log_2(a \log(N/\varepsilon) + b)$ and the depth is $O(\log \log(N/\varepsilon))$.

The blocking over s sites makes the two-site injectivity assumed for the multiscale network of [con26d] unnecessary here: the tree is that of the blocked tensor, whose sites are the registers. The formal statements¹ are exact, and no outcome is post-selected.

¹Formal declarations: `QuantumCircuit.TeleportHop.isImplementationOn_round` in `TNLean/Circuit/Teleportation/Round`, `QuantumCircuit.LongRangeGate.isRoundsImplementationOn_rounds` in `TNLean/Circuit/Teleportation/LongRangeGates`, and `MPSPreparation.isPreparedWithMeasurementRoundsInDepth_treeOp` in `TNLean/MPS/Preparation/TreeMeasurement`, `QuantumCircuit.RegisterGate.exists_rounds` in `TNLean/Circuit/Teleportation/RegisterGates`, `MPSPreparation.exists_rounds_blockLayerOp_regTreeOp` in `TNLean/MPS/Preparation/RegisterTree`, `MPSPreparation.regTreeOp_apply_blockInputCfg` in `TNLean/MPS/Preparation/RegisterTreeState`, and `MPSPreparation.exists_isPreparedWithMeasurementRoundsInDepth_approximatingMPVState`, `MPSPrepara`

3 The restriction formerly adopted: the chain length

The block length $q = s2^{k+1}$ divided the chain length N , so that the N/q blocks of eq. (10) all had q sites and every tree was complete. The depth bound held for the chain lengths N divisible by $s2^{k+1}$ for some k with $a \log(N/\varepsilon) + b \leq s2^{k+1} \leq 2(a \log(N/\varepsilon) + b)$, for instance $N = Ms2^K$ with K large, and not for every chain length; since the block length $s2^{k+1}$ is even, no odd N was covered. The source states the depth for every N .

4 Elimination plan

The depth bound for every chain length of the circuit without measurements uses unequal blocks: the last block has length q' with $q \leq q' < 2q$ [con26c]. A complete binary tree of registers covers $s2^{k+1}$ sites, so the last block needs a tree whose leaves have unequal lengths; each node is still a unitary on a bounded number of sites, so the depth stays $O(k)$.

5 Resolution

The restriction on the chain length is removed by trees on leaves of unequal widths.²

1. *Halves of unequal lengths.* For polar decompositions $\mathcal{B}_{n_i} = V_{n_i} P_{n_i}$ of A blocked over n_1 and over n_2 sites, $V_{n_1+n_2} = (V_{n_1} \otimes V_{n_2})W$, where W is the partial isometry of the polar decomposition of the tensor $(a, b) \mapsto P_{n_1}^a P_{n_2}^b$ of two neighbouring sites. No injectivity is needed: the initial projector of $V_{n_1} \otimes V_{n_2}$ fixes that tensor, so $V_{n_1} \otimes V_{n_2}$ composes with its partial isometry. The tree below uses injectivity only to make W and the leaf maps isometries.
2. *The tree of a block.* A block of $\ell \geq 2^{h+2}s$ sites is cut into 2^{h+1} leaves of even widths, as equal as possible, at least $2s$ each; when ℓ is odd the last leaf has one more site. The node p of depth $j \leq h$ covers 2^{h+1-j} consecutive leaves, its registers are its first s sites and the last s sites of its

`tion.exists_isPreparedWithMeasurementRoundsInDepth_approximationError_le` and `MPSPreparation.exists_isPreparedWithMeasurementRoundsInDepth_le_logb_log` in `TNLean/MPS/Preparation/LogLogDepthPreparation`.

²Formal declarations: `MPSPreparation.cfgPolarIso_split` in `TNLean/MPS/Preparation/PolarMerge`, `MPSPreparation.exists_rounds_blockLayerOp_treeLevelsOp` in `TNLean/MPS/Preparation/UnequalRegisterTree`, `MPSPreparation.treeBlockOp_apply` in `TNLean/MPS/Preparation/UnequalRegisterTreeState`, `MPSPreparation.exists_isPreparedWithMeasurementRoundsInDepth_blockIsometryState` in `TNLean/MPS/Preparation/UnequalTreePreparation`, and `MPSPreparation.exists_isPreparedWithMeasurementRoundsInDepth_le_log_log_of_mvState_ne_zero` and `MPSPreparation.exists_isPreparedWithMeasurementRoundsInDepth_le_log_log` in `TNLean/MPS/Preparation/LogLogDepthEveryLength`.

last leaf, not counting the extra site, and its unitary extends the isometry W joining its two halves. The leaves carry unitaries extending $V_{\ell'}$ on their ℓ' sites. The widths are even, so the two registers of every node are an even number of sites apart, as the register gates require. The tree implements V_ℓ on the input of the block.

3. *The legs of the pairs.* The register length is $s = L+2$, with L the injectivity length of the tensor in the gauge of eq. (5), so that $D \leq d^{s-1}$ and a leg of a pair is written on the first $s-1$ sites of a register, the last site in $|0\rangle$. For odd ℓ the right leg of the pair of a block then ends on the extra site, which carries $|0\rangle$, and the input of the block lies on the two registers of the root.
4. *The depth.* The depths $0, \dots, h$ of the trees of all the blocks are applied one after the other with measurements, and the leaves by one local circuit; for blocks of lengths between $2^{h+2}s$ and $2^{h+1}c$ the depth is at most $C(h+1)$, with C depending on d , s and c .
5. *Every chain length.* With $q = \lceil a \log(N/\varepsilon) + b \rceil$ and $q \leq N$, the chain is cut into $M-1$ blocks of q sites and one of $q' = q + (N \bmod q) < 2q$ sites, Lemma 1'(i) for unequal blocks bounds the error by ε , and $2^{h+2}s \leq q < 2^{h+3}s$ gives the depth $C(h+1) \leq c \log(\log(N/\varepsilon) + 1)$. A chain with $N < q$ is one block, and $|\phi_N\rangle$ itself is the state of that block with the pair $\omega \propto P|\mathbb{K}\rangle$; below $4s$ sites it is prepared in bounded depth.

The main theorem holds for every $N \geq 2$ and $0 < \varepsilon \leq 1$ with $|\phi_N(A)\rangle \neq 0$, the presupposition recorded in `docs/paper-gaps/mswc24_depth_upper_bound_n_onzero_state.tex`, and for every $N \geq N_0$. The bound $c \log(\log(N/\varepsilon) + 1)$ is $O(\log \log(N/\varepsilon))$; the shift by 1 keeps it positive for $N/\varepsilon \geq 2$.

6 Supported polar trees

For a tensor injective on a common set S of virtual pairs after a fixed length, choose a coordinate inclusion $J: \mathbb{C}^\chi \rightarrow \mathbb{C}^{D^2}$ onto S . The restricted factors $\widehat{V}_n = V_n J$ are isometries and obey $\widehat{V}_{l+r} = (\widehat{V}_l \otimes \widehat{V}_r) \widehat{W}_{l,r}$, where $\widehat{W}_{l,r} = (J^\dagger \otimes J^\dagger) W_{l,r} J$ satisfies $\widehat{W}_{l,r}^\dagger \widehat{W}_{l,r} = I$. The bounded-leaf tree and its measurement implementation now apply with arbitrary virtual dimension χ . They preserve all supported superpositions simultaneously. For each fixed measurement history the branch scalar is independent of the input on the zero-scratch subspace; the final equality specifies every physical site.³ The supported trees now compose with the constant-depth sparse GHZ seed and one simultaneous pair-window unitary. The bond encodings end in a zero site, and an equality of full physical

³Formal declarations: `MPSPreparation.exists_isometryTree_matrix_eq_supportedCfPolarIso` in `TNLean/MPS/Preparation/SupportedPolarTree`, and `MPSPreparation.exists_rounds_blockLayerOp_supportedPolarTree` in `TNLean/MPS/Preparation/SupportedTreePreparation`.

configurations aligns them with the even tree-root cutoff also for odd block lengths.

7 Canonical non-normal resolution

The all-length bound now holds for the supplied canonical repeated-block class with normal inequivalent left-canonical blocks and positive definite trace-one transfer fixed points.⁴ The same constant precedes N and ε . It covers every $N \geq 2$ with nonzero periodic vector and $0 < \varepsilon \leq 1$, finite-block overlap, complex copy phases, repetitions, and vanishing individual sector sums. The approximation uses the unweighted direct sum, and its common rate bounds the mixed maps as well as the individual maps, as in the inherited corrections of [con26b].

For $q = \lceil a \log(N/\varepsilon) + b \rceil \leq N$, use the unequal blocks and the amplitude-uniform exponential error estimate. If $N < q$ and $N \geq 4s$, use a whole-ring supported tree of height $O(\log N)$. The exact one-block pair is required to have support only at $M = 1$; it need not have that support at other ring sizes. Only the finitely many lengths $N < 4s$ use bounded exact synthesis. Thus no accuracy-dependent linear-depth fallback or free N -site gate is hidden in the bound.

Nearest-neighbour unitary layers are counted. Measurement, classical communication and on-site correction are free, and the number of rounds may grow with the tree height. Every nonzero-probability history gives the target up to a scalar, with no postselection or discarded environment. The original-to-canonical transport for arbitrary tensors and their periodicity data is not asserted here.

8 The remaining restriction

A two-site gate between distant sites joins sites at odd separation, on pairwise disjoint stretches; gates at even separation, or on overlapping stretches, are not covered, whereas the paragraph of the source speaks of teleporting spatially separated sites in general. Neither the tree of two-site gates nor the trees of registers need such gates. This geometric limitation does not restrict the canonical non-normal construction just described. The earlier logarithmic-depth construction and its corrections are recorded in [con26a].

⁴Formal declarations: `MPSPreparation.exists_supported_block_preparation_constants` in `TNLean/MPS/Preparation/CoherentBlockTreePreparation`, `MPSPreparation.exists_log_log_preparation_of_supported_block_approximation` in `TNLean/MPS/Preparation/SupportedLogLogPreparation`, and `MPSPreparation.exists_isPreparedWithMeasurementRoundsInDepth_repeatedBlockSum_le_log_log_of_inequivalent` in `TNLean/MPS/Preparation/OverlapLogLogPreparation`.

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