

Sequential Factorization of the Blocked Isometry: Positive Block Length

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Verdict: scope-restriction (open).

1 What the source states

Let A be a tensor of physical dimension d and bond dimension D , and let $\mathcal{B}_q: \mathbb{C}^D \otimes \mathbb{C}^D \rightarrow (\mathbb{C}^d)^{\otimes q}$ be its map blocked over q sites, with polar decomposition $\mathcal{B}_q = VP$. Equations (13)–(15) of [MSWC24] write the isometry as an ordered product of q isometries acting on one site and a bond each,

$$V = V_q \cdots V_1, \tag{1}$$

where $V_k: \mathbb{C}^{D'_k} \rightarrow \mathbb{C}^{D'_{k+1}} \otimes \mathbb{C}^d$, $D'_1 = D^2$, $D'_{q+1} = 1$ and $D'_k \leq D^2$. The source states no lower bound on the block length q ; the construction factors the blocked map site by site and blocks at least one site.

2 The empty block

For $q = 0$ the blocked map has a single matrix, the empty product $\mathbb{1}_D$, and the output space $(\mathbb{C}^d)^{\otimes 0}$ is $\mathbb{C}$. The one matrix $\mathbb{1}_D$ spans the full matrix algebra only when $D \leq 1$. When $D = 1$, $V = 1$ and (1) is the empty product with $D'_1 = D'_{q+1} = 1$, so the statement holds trivially. When $D = 0$ the space $\mathbb{C}^D \otimes \mathbb{C}^D$ is zero, and the requirement $D'_1 = D^2 = 0$ contradicts $D'_{q+1} = 1$ at $q = 0$, where these are the same bond. The empty block carries no content of the preparation scheme, which blocks at least one site.

3 The restriction adopted

The sequential factorization is stated for block lengths $q \geq 1$ and injective $\mathcal{B}_q$.¹ The same bound $q \geq 1$ is assumed by the variant for a blocked map injective

¹Formal declaration: `MPSPreparation.exists_isometric_chain_polarIsoMatrix` in `TNLean/MPS/Preparation/SequentialFactorization`.

on a set S of bond pairs, whose isometric factor is factored on the inputs of S , with last bond $D'_1 = |S|$.² At $q = 0$ the map $\mathbb{1}_D$ vanishes outside S and spans the matrix units in S only when $D = 0$ and S is empty, or $D = 1$ and S is the one pair; the conclusion requires $|S| = 1$ and fails in the first case. The bound $q \geq 1$ is also assumed by the variant with no injectivity, which factors the partial isometry V composed with a unitary T on $\mathbb{C}^{D^2}$ whose first r columns span the range of $V^\dagger V$, with last bond $D'_1 = r$.³ At $q = 0$ this conclusion fails for every D . When $D = 0$, $r \leq D^2 = 0$ contradicts $D'_1 = D'_{q+1} = 1$. When $D \geq 1$, $V^\dagger V$ is the projector onto the line of $\mathbb{1}_D$, so $r = 1$, and the conclusion asks that the scalar $VT|0\rangle$ equal the empty product 1. This scalar has modulus one, and multiplying the first column of T by -1 keeps the hypotheses and changes its sign, so the conclusion fails for one of the two unitaries. For every $q \geq 1$ the conclusion is exactly (1) in matrix product form, with no further hypothesis. Elimination for the two injective variants: replace $q \geq 1$ by a positive bond dimension $D \geq 1$, which the source takes for granted, and treat the case $q = 0$, $D = 1$ directly as the empty product. This does not cover the variant with no injectivity, whose conclusion fails at $q = 0$ for every $D \geq 1$. Its restriction can be removed only by changing the statement: either let the boundary carry a phase, concluding $\langle \sigma | VT | x \rangle = c(Q_0(\sigma_0) \cdots Q_{q-1}(\sigma_{q-1}))_{0x}$ for a scalar c of modulus one, or keep $q \geq 1$ and state the empty block separately as the scalar $VT|0\rangle$ of modulus one. Since the source blocks at least one site, the second option is the planned one.

References

- [MSWC24] Daniel Malz, Georgios Styliaris, Zhi-Yuan Wei, and J. Ignacio Cirac. Preparation of matrix product states with log-depth quantum circuits. *Phys. Rev. Lett.*, 132:040404, 2024.

²Formal declaration: `MPSPreparation.exists_isometric_chain_polarIsoMatrix_of_isInjectiveOn` in the same module.

³Formal declaration: `MPSPreparation.exists_isometric_chain_polarIsoMatrix_mul_unitary` in the same module.