

Repeated Overlapping Blocks with a Small Weight Sum

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Verdict: scope-restriction (resolved).

1 What the source states

Let $A^i = \bigoplus_{j=1}^b \text{diag}(\mu_{j,1}, \dots, \mu_{j,m_j}) \otimes A_j^i$ with normal A_j [MSWC24, eq. (S2)]. The source assumes “Without loss of generality we assume $|\mu_{j,k}| \leq 1$ with at least one of them having magnitude exactly one” [MSWC24, after eq. (S2)], puts $\beta_j = \sum_k \mu_{j,k}^N$ [MSWC24, eq. (S4)], and bounds the error $\varepsilon = 1 - |\langle \phi_N | \phi_N \rangle|$ of its approximating state by $O((N/q)e^{-\gamma q/\xi_{\text{diag}}})$ [MSWC24, Lemma 1'(ii)], with no factor depending on the weights.

2 The former restriction

For blocks with multiplicities whose q -site states may overlap, the bound for the corrected approximating state of [con26a], at the rate of [con26c], was first proved for complex weights with $|\mu_{j,k}| \leq 1$ in the form

$$\varepsilon \leq \frac{C y e^{C y}}{\min(1, b)^{1/2}}, \quad y = M e^{-\gamma q/\xi}, \quad b = \sum_j |\beta_j|^2,$$

and without the factor only for real weights $0 \leq \mu_{j,k} \leq 1$ of which one equals one, where $b \geq 1$. Within the source’s normalization $b < 1$ can occur: for copy weights $(1, -1)$ at odd N the sum β_j of that block vanishes, and b is then the contribution of the other blocks, which may be exponentially small in N . The factor came from the proof: the overlaps $z_j = \langle \Omega_j | \phi_M(L_j^\dagger P) \rangle$ of the rows of the positive part P with the fixed-point tensors of the blocks were estimated in absolute terms, $|z_j - \beta_j| \leq C_j y e^{C_j y}$, and dividing by $b^{1/2}$ to normalize lost the factor.

3 The relative estimate

The factor is removed by estimating the positive part relative to the weights of the blocks. With the copy norms $c_j = (\sum_k |\mu_{j,k}|^{2q})^{1/2}$ and the copy isometries L_j of [con26a], the blocked map is $B = \tilde{B}C$ with $\tilde{B} = \sum_j B_j L_j^\dagger$ and $C = \sum_j c_j L_j L_j^\dagger$. Put $C^{-1} = \sum_j c_j^{-1} L_j L_j^\dagger$, $P_\infty = \sum_j c_j L_j (\sqrt{\sigma_j}^T \otimes \mathbf{1}) L_j^\dagger$ and $\Delta = \tilde{B}^\dagger \tilde{B} - \sum_j L_j (\sigma_j^T \otimes \mathbf{1}) L_j^\dagger$. Then $X = (P - P_\infty)C^{-1}$ solves the Sylvester equation

$$PX + XP_\infty = C\Delta\Pi, \quad \Pi = \sum_j L_j L_j^\dagger.$$

If $\lambda > 0$ bounds every $\sigma_j^T \otimes \mathbf{1}$ from below and $\|\Delta\| \leq \lambda/2$, then $P^2 = C\tilde{B}^\dagger \tilde{B}C \geq (\lambda/2)C^2$. In eigenbases $P = \sum_a p_a f_a f_a^\dagger$ and $P_\infty = \sum_i s_i e_i e_i^\dagger$ the equation reads $(p_a + s_i)\langle f_a | X e_i \rangle = \langle C f_a | \Delta \Pi e_i \rangle$, and $\|C f_a\| \leq (\lambda/2)^{-1/2} p_a$, so every entry of X has modulus at most $(\lambda/2)^{-1/2} \|\Delta\|$; the entries with $p_a = s_i = 0$ vanish because $X = PC^{-1} - C^{-1}P_\infty$. This is the Schur-product argument of the relative perturbation bounds for the unitary polar factor [Li97]. The Gram estimate of [con26c] at unit weights gives $\|\Delta\| \leq K e^{-\gamma q/\xi}$ for every $0 < \gamma < 1$, independently of the weights, so for all q beyond a threshold

$$\|(P - P_\infty)C^{-1}\| \leq K e^{-\gamma q/\xi},$$

uniformly in the nonzero weights, which need not satisfy $|\mu_{j,k}| \leq 1$. The blocks $L_j^\dagger P C^{-1} L_{j'}$ are therefore $\delta_{jj'} (\sqrt{\sigma_j}^T \otimes \mathbf{1})$ up to $K e^{-\gamma q/\xi}$.

The row $L_j^\dagger P$, read as a tensor, is the direct sum over the copies (j', k) with the weights $\mu_{j',k}^q$ of the tensors read from the blocks $L_j^\dagger P C^{-1} L_{j'}$, so

$$z_j = \sum_{j'} \beta_{j'} w_{jj'}, \quad w_{jj'} = \langle \Omega_j | \phi_M(L_j^\dagger P C^{-1} L_{j'}) \rangle,$$

and the telescoping estimate gives $|w_{jj'} - \delta_{jj'}| \leq C_{jj'} y e^{C_{jj'} y}$. Hence

$$\frac{1}{b} \sum_j \bar{\beta}_j z_j - 1 = \frac{1}{b} \sum_{j,j'} \bar{\beta}_j \beta_{j'} (w_{jj'} - \delta_{jj'})$$

has modulus at most $\sum_{j,j'} C_{jj'} y e^{C_{jj'} y}$, since $|\beta_j \beta_{j'}| \leq b$, and the rest of the proof of [con26a] goes through with no division by $b^{1/2}$.

4 What is formalized

For blocks with multiplicities whose q -site states may overlap, normal in the gauge $\sum_i (A_j^i)^\dagger A_j^i = \mathbf{1}$ with positive definite fixed points σ_j of trace one, with λ_2 bounding the moduli of the eigenvalues other than one of every transfer matrix of a block and of all eigenvalues of the mixed transfer matrices of distinct blocks,

and $0 < \gamma < 1$, there are $C, C' > 0$ such that for all nonzero complex weights, every q and every $M \geq 1$ with $\beta \neq 0$, the corrected approximating state satisfies

$$\varepsilon \leq Cy e^{Cy}, \quad \varepsilon \leq C'y, \quad y = Me^{-\gamma q/\xi}.$$

The constants do not depend on the weights, and the normalization $|\mu_{j,k}| \leq 1$ is not used. The same bounds hold for the source's state with the pairs on one copy of each block and the corrected coefficients of [con26b].¹ The remaining deviation from the source is the rate, recorded in [con26c].

5 Numerical check

A script evaluates the corrected state in arithmetic of 90 significant digits for the two random normal blocks of bond dimension two on $d = 2$ used in [con26a], with $|\lambda_2| = 0.850$, at $q = 5, 9, 13$, where $N = qM$ is odd for odd M .

- For copy weights $(1, i, -1, -i)$ on the first block and 0.5 on the second, $\beta_1 = 0$ whenever $4 \nmid N$. At $M = 5$ the error is $0.427, 7.55 \cdot 10^{-2}, 1.51 \cdot 10^{-2}$, while $b = 0.5^{2N}$ is $8.9 \cdot 10^{-16}, 8.1 \cdot 10^{-28}, 7.3 \cdot 10^{-40}$; $\varepsilon/(M|\lambda_2|^q)$ is $0.193, 6.53 \cdot 10^{-2}, 2.51 \cdot 10^{-2}$.
- For copy weights $(1, -1)$ and 0.5, at $M = 1, 3, 5$ the ratio $\varepsilon/(M|\lambda_2|^q)$ stays between $1.78 \cdot 10^{-2}$ and 0.227 over the three values of q , while b ranges from $9.8 \cdot 10^{-4}$ down to $7.3 \cdot 10^{-40}$.

The error is controlled by y alone, with no growth as b decreases, as the bound states.

References

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¹Formal declarations: `Matrix.norm_le_of_mul_add_mul_eq_mul` in `TNLean/Algebra/MatrixSylvesterBound`; `MPSTensor.norm_polarPos_sub_mul_sum_le` and `MPSTensor.exists_norm_relativeRow_sub_le` in `TNLean/MPS/Preparation/RelativePositivePart`; `MPSTensor.mpvOverlap_conjTranspose_copyIsometry_mul_polarPos`, `MPSTensor.exists_norm_mpvOverlap_sub_ite_le`, `MPSTensor.exists_approximationError_le_repeatedOverlappingBlockSum` and `MPSTensor.exists_approximationError_le_mul_repeatedOverlappingBlockSum` in `TNLean/MPS/Preparation/RepeatedOverlappingBlockWeights`; `MPSTensor.exists_approximationError_le_repeatedOverlappingBlockSum_oneCopy` and `MPSTensor.exists_approximationError_le_mul_repeatedOverlappingBlockSum_oneCopy` in `TNLean/MPS/Preparation/OneCopyPairState`.

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